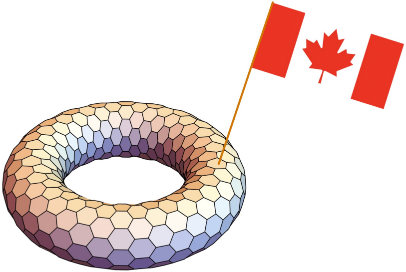
Canonical decompositions of 3-connected graphs

Jan Kurkofka

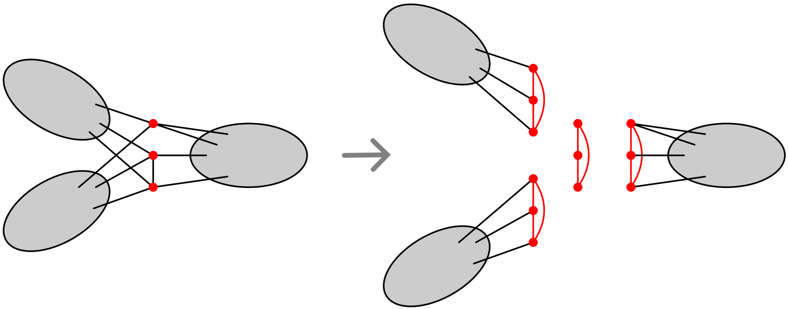


Joint work with Johannes Carmesin

University of Birmingham

Problem: Decompose k-con'd G along k-separators into pieces that are $(k+1)$-con'd or 'basic'.

Decomposing G along a k-separator:



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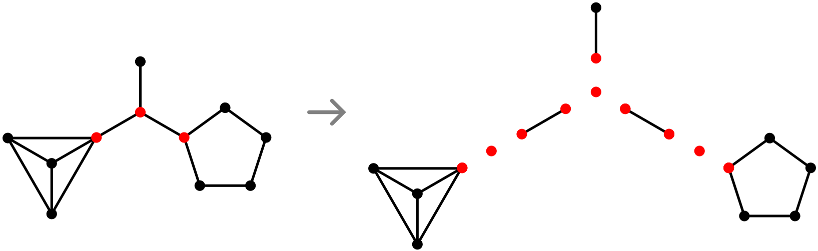

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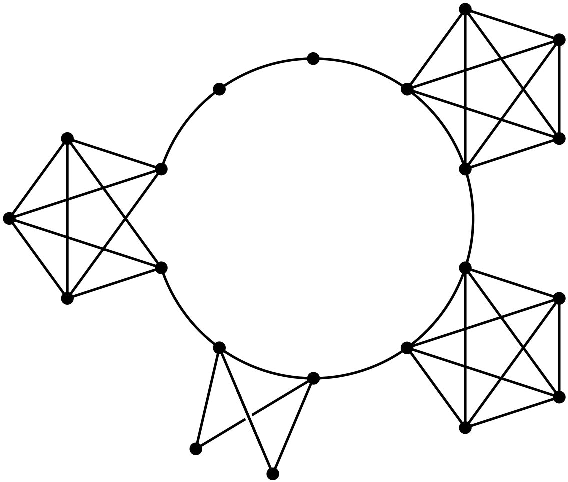

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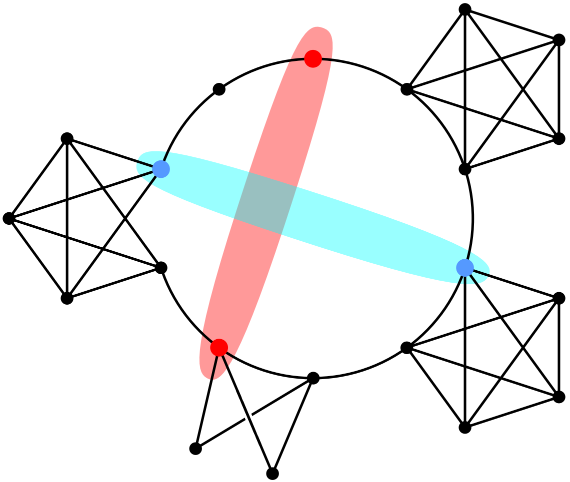

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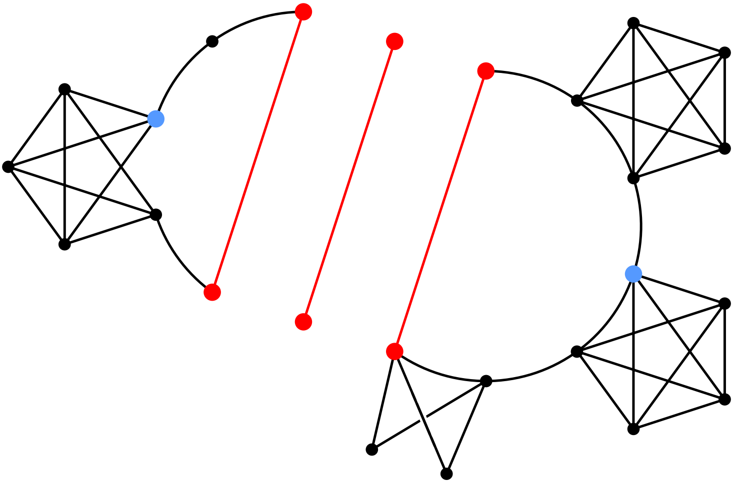
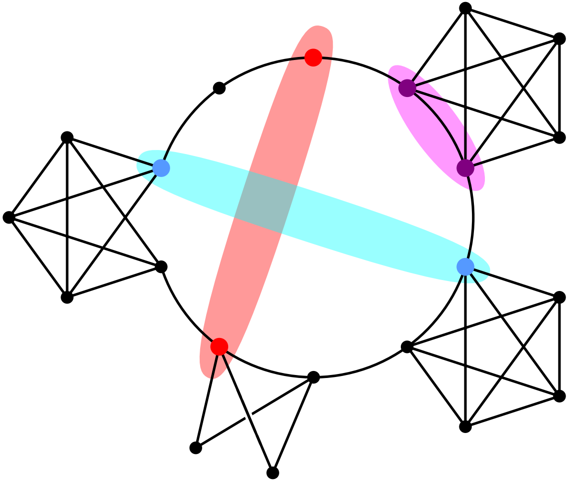


Two k-separators are *nested* if neither separates the other; otherwise they *cross*.



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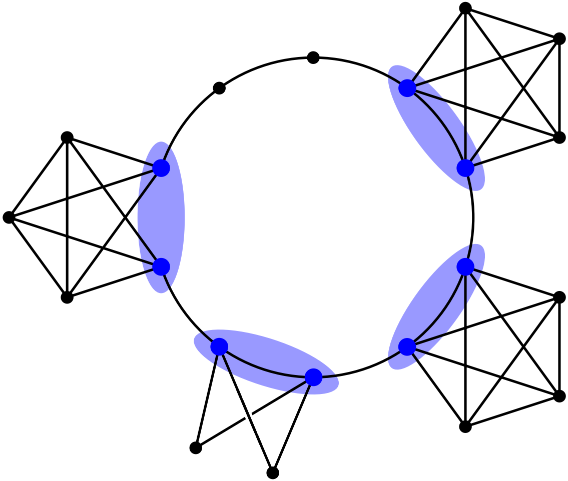

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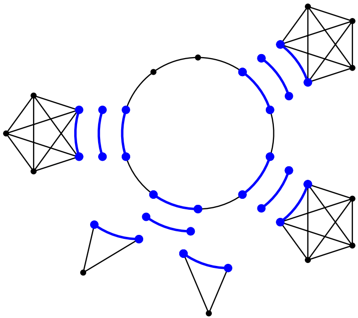
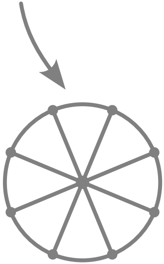


Theorem (Cunningham & Edmonds 80)

Every 2-con'd G decomposes along its totally-nested 2-separators into 3-con'd graphs, cycles and K_2's.

Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into 4-con'd graphs, wheels and K_3's.



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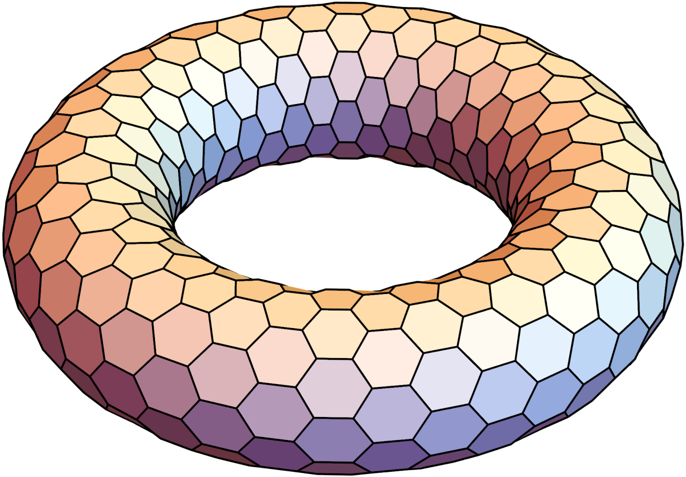

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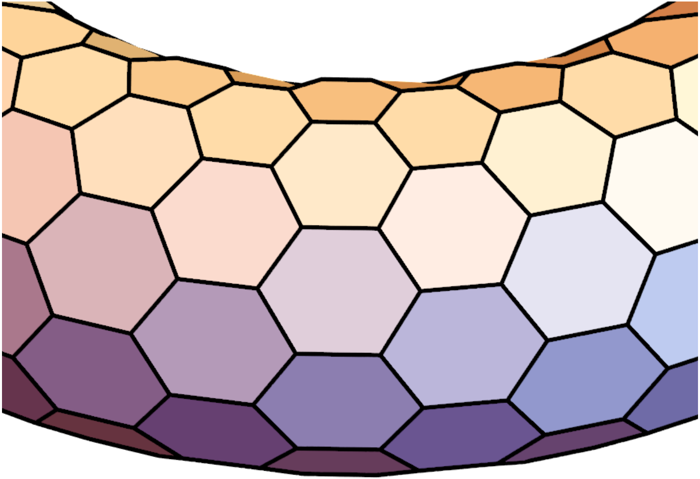

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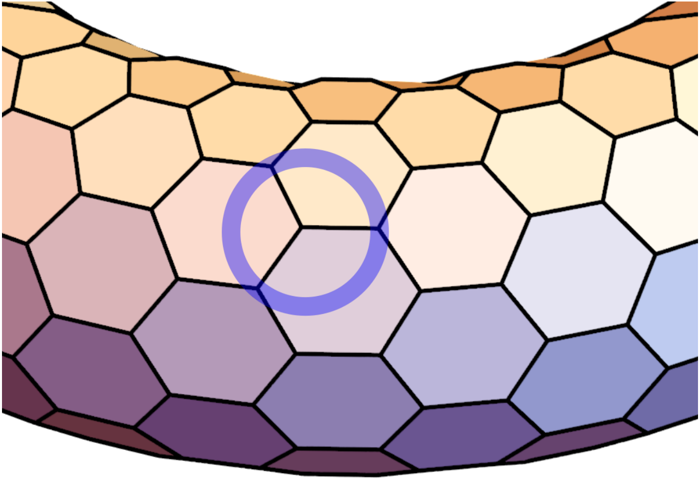

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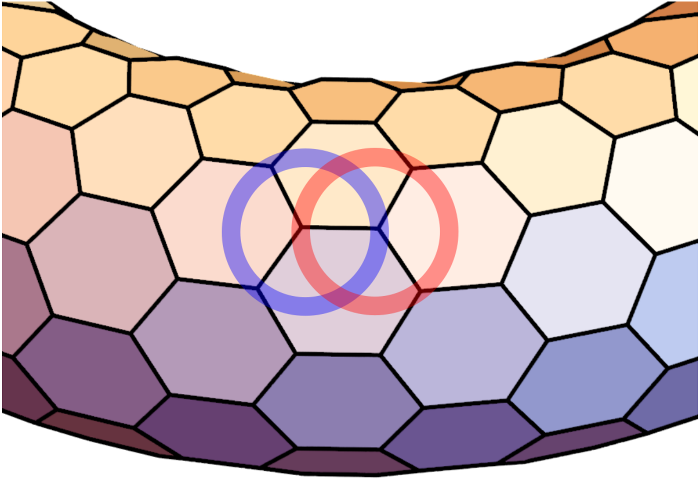
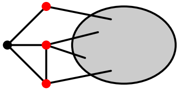
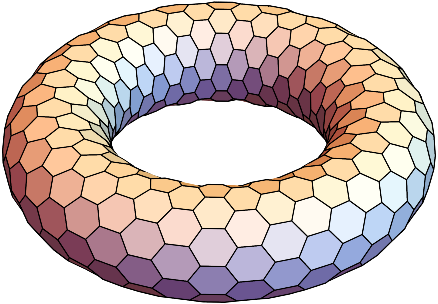


Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into quasi 4-con'd graphs, wheels and K_3's.

↓

3-con'd, > 4 vertices, every 3-separator has form 



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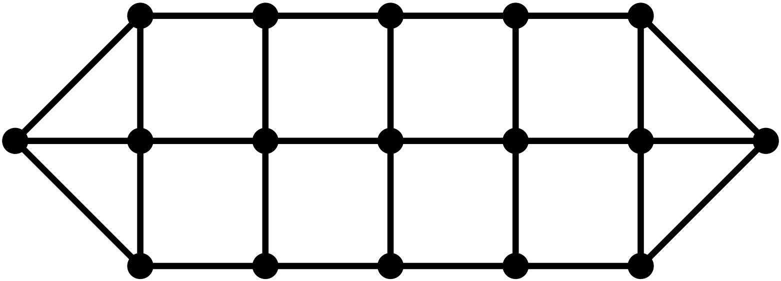

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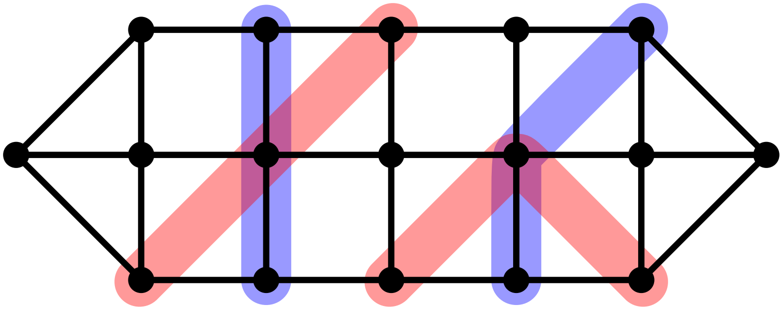

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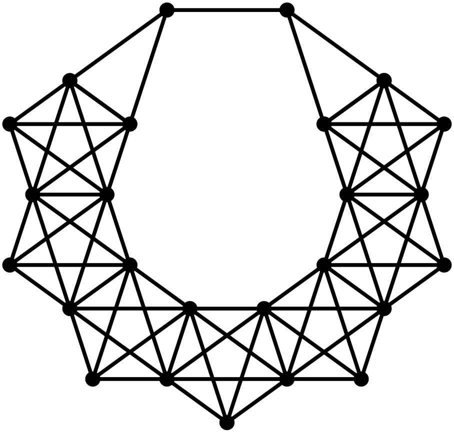

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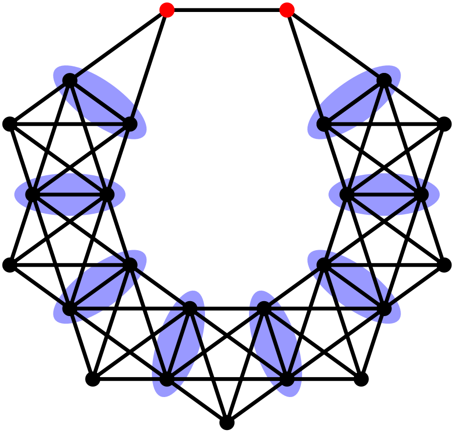

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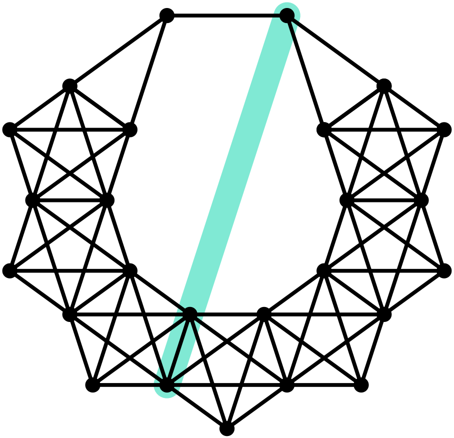

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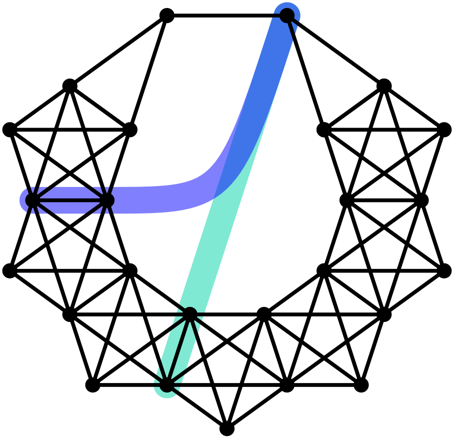

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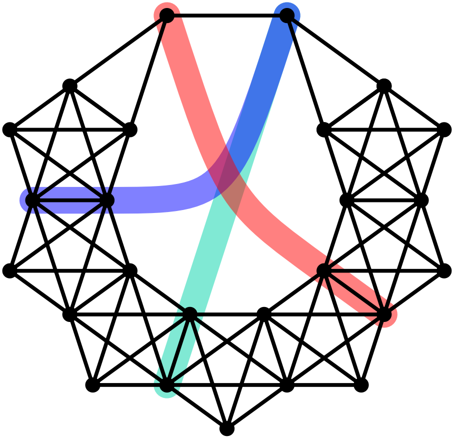

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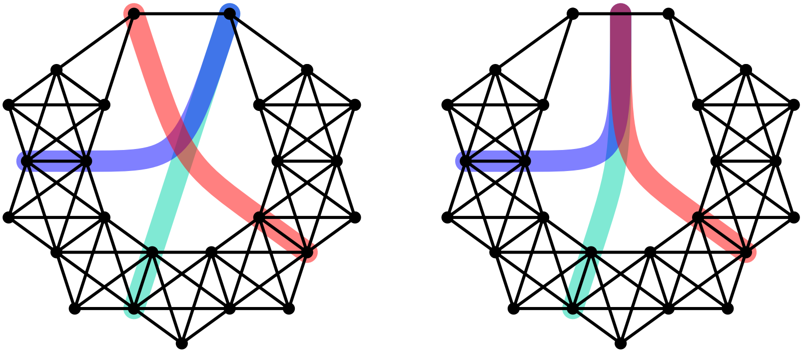

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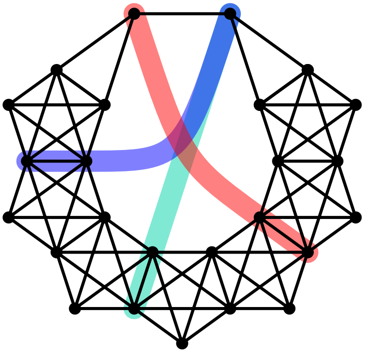

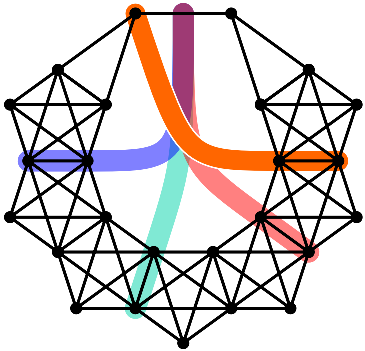

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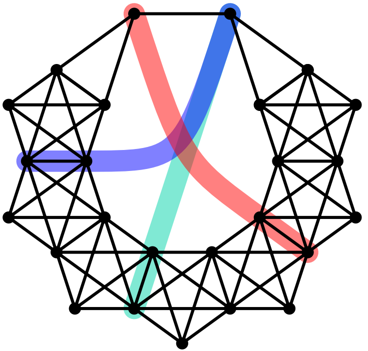

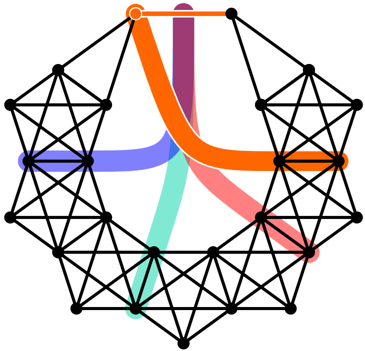

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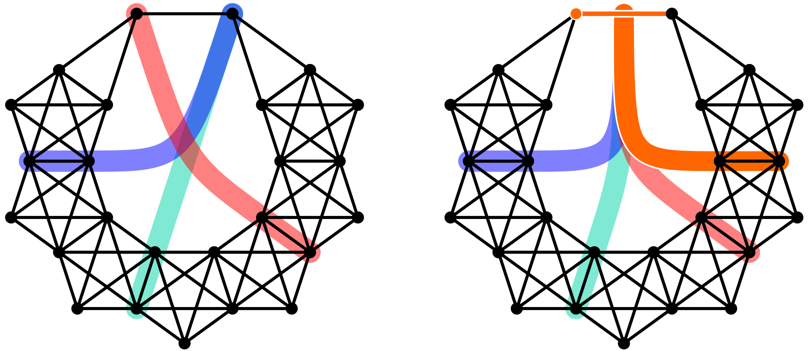
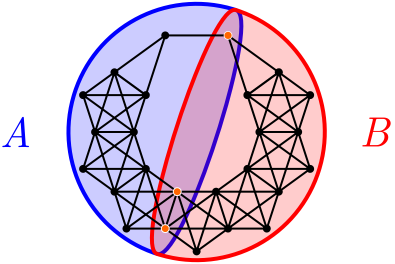
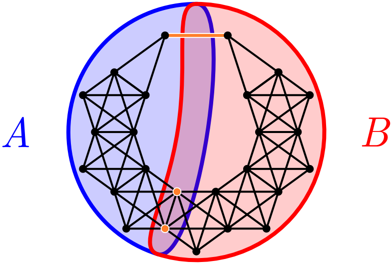




separation of G: $\{A, B\}$ with $A \cup B = V(G)$ and $E(A \setminus B, B \setminus A) = \emptyset$

separator of $\{A, B\}$: $A \cap B$



mixed-separation of G: $\{A, B\}$ with $A \cup B = V(G)$ and

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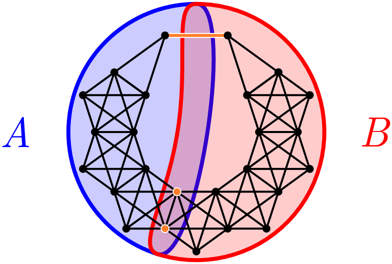

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tri-separation of G: mixed-sep'n $\{A, B\}$ with $|\text{sep'r}| = 3$
and every vx in $A \cap B$ has two neighb's
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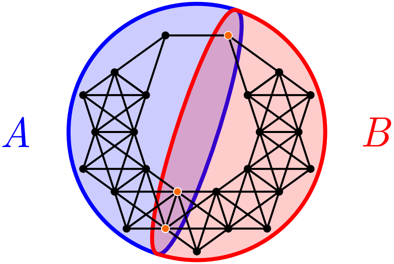

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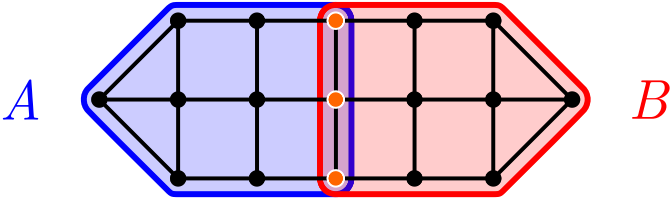

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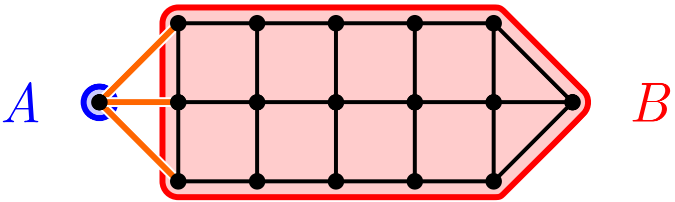
trivial



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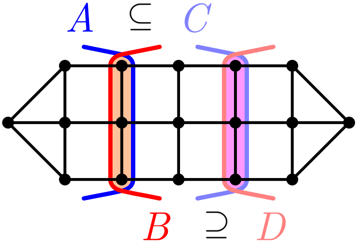
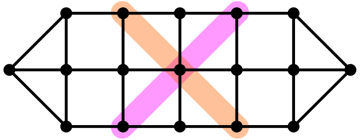
$\{A, B\}$ and $\{C, D\}$ are *nested* if $A \subseteq C$ and $B \supseteq D$ after possibly switching A with B or C with D; otherwise they *cross*.

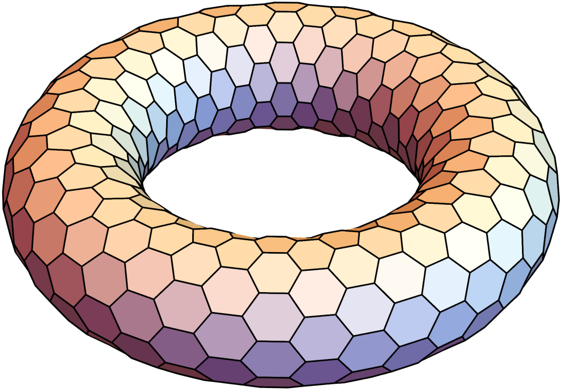
nested



crossing



totally-nested nontrivial tri-separations



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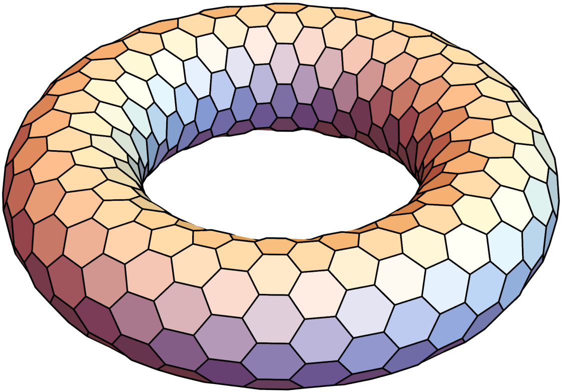

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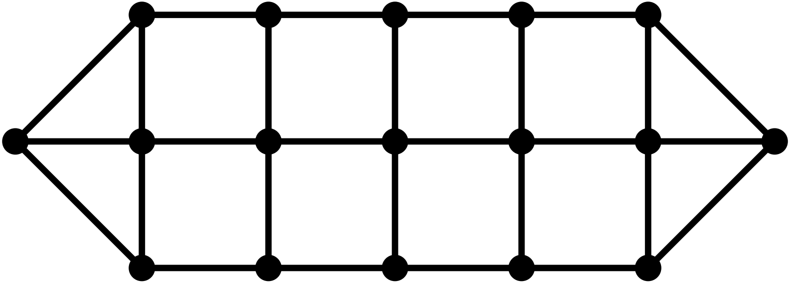

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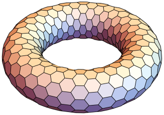

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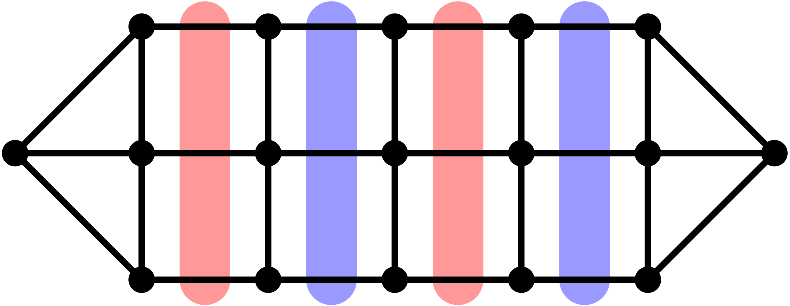

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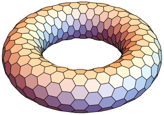

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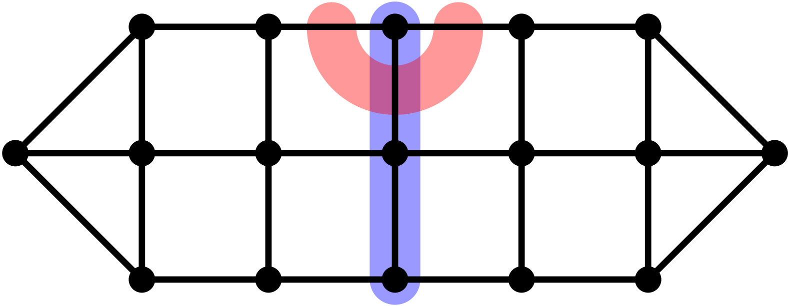

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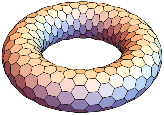

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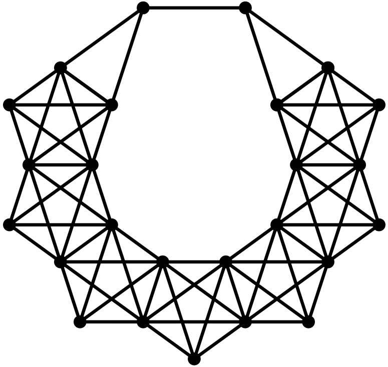

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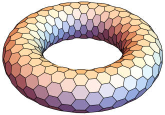

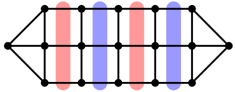

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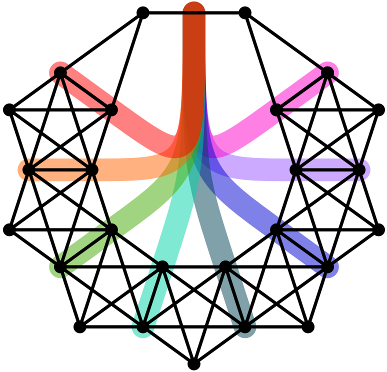

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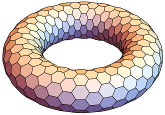
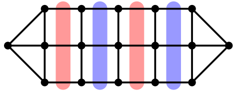
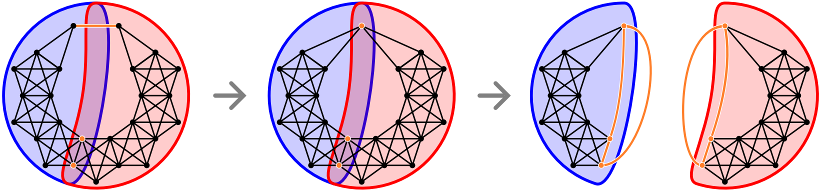
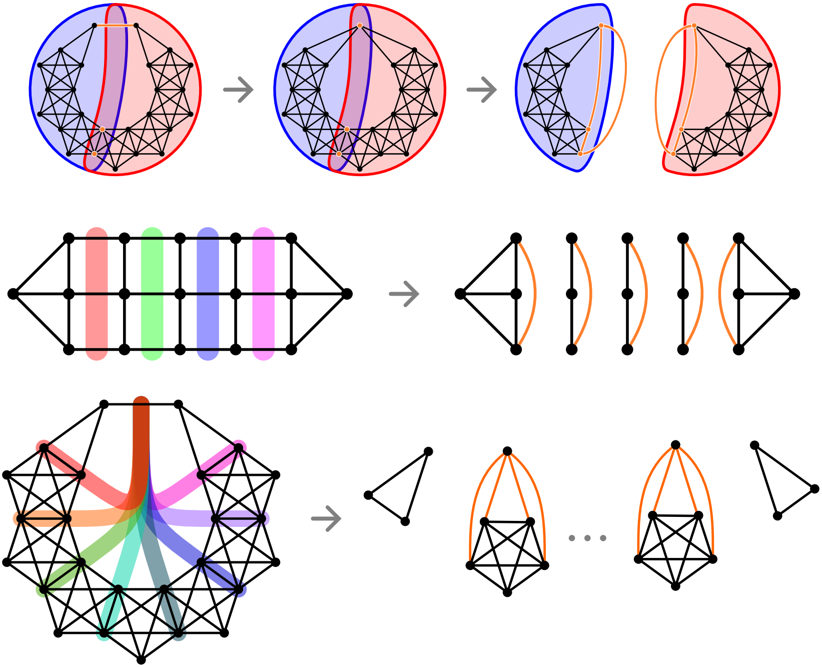




Decomposing along a tri-separation



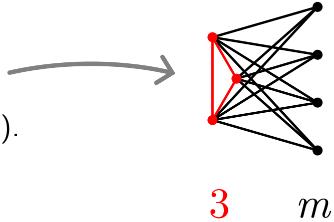


Main result (Carmesin & K. 23)

Every 3-con'd G decomposes along its totally-nested nontrivial tri-separations into minors of G that are

- quasi 4-con'd
- wheels
- thickened $K_{3,m}$

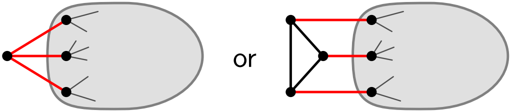
or $G = K_{3,m}$ $(m \geqslant 0)$.



Application 1 (Carmesin & K. 23)

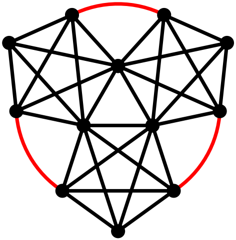
Every vertex-transitive finite con'd G is either

- 4-con'd
- 3-con'd and 3-regular and every tri-sep'n has form



- $K_1, \ldots, K_4$ or a cycle.

Application 2: Connectivity Augmentation to 4



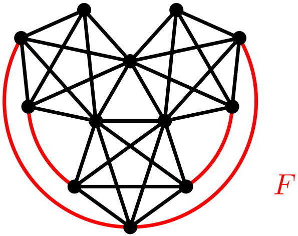
Theorem (Carmesin & Ramanujan 23+)

$\exists$ FPT-algorithm with runtime $C(k) \cdot \mathrm{Poly}(\,|V(G)|\,)$ and

Input: Graph G, $k \in \mathbb{N}$ and $F \subseteq E(\,\overline{G}\,)$

Output: No, or $\leqslant k$-sized $X \subseteq F$ such that $G + X$ is 4-con'd

Application 2: Connectivity Augmentation to 4



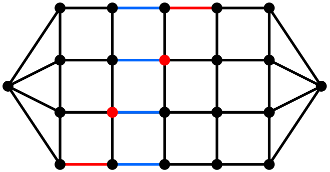
Theorem (Carmesin & Ramanujan 23+)

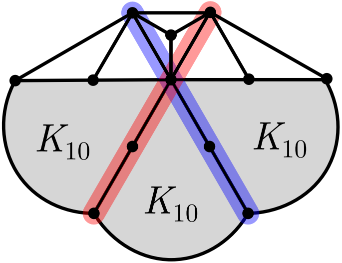
$\exists$ FPT-algorithm with runtime $C(k) \cdot \mathrm{Poly}(\,|V(G)|\,)$ and

Input: Graph G, $k \in \mathbb{N}$ and $F \subseteq E(\,\overline{G}\,)$

Output: No, or $\leqslant k$-sized $X \subseteq F$ such that $G + X$ is 4-con'd

Open: Extend the main result to k-separations for $k \geqslant 4$.





Open: Tri-separations for matroids

$k = 2$:	✓ finite	Cunningham & Edmonds 80
	✓ infinite	Aigner-Horev, Diestel & Postle 16
$k = 3$:	???	Related: Oxley, Semple & Whittle 04

Tri-separation

Mixed-sep'n $\{A, B\}$ with $|$sep'r$| = 3$ such that every vx in $A \cap B$ has two neighb's in $G[A]$ and in $G[B]$.

Main result (Carmesin & K. 23)

Every 3-con'd G decomposes along its totally-nested nontrivial tri-separations into minors of G that are quasi 4-con'd, wheels, thickened $K_{3,m}$'s or $G = K_{3,m}$ $(m \geqslant 0)$.

Open

Extend to k-separations for $k \geqslant 4$. Tri-separations for matroids.

arXiv: 2304.00945 Slides: web.mat.bham.ac.uk/J.Kurkofka/

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Thank you!