Towards a Stallings-type theorem for finite groups

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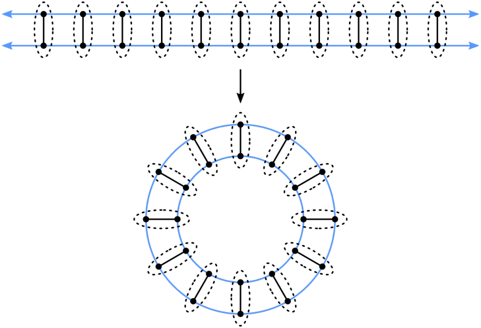


Joint work with

Johannes Carmesin, George Kontogeorgiou and Will J. Turner

Theorem (Stallings).

TFAE for every group Γ with finite generating set S:

- Cay(Γ, S) has $\geqslant 2$ ends;
- Γ decomposes as a non-trivial amalgamated free product or HNN-extension over a finite subgroup.

(This is independent of S.)

end of a graph: equivalence class of one-way infinite paths w.r.t. the relation 'not separable by finitely many vertices'

Open problem: Extend Stallings' theorem to finite groups.

Challenges:

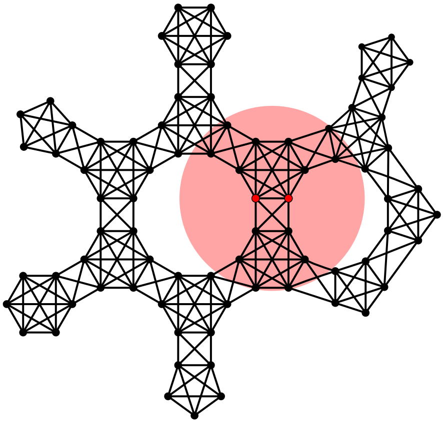
1. Ends have no finite counterparts
2. Key step of the proof fails for finite Γ

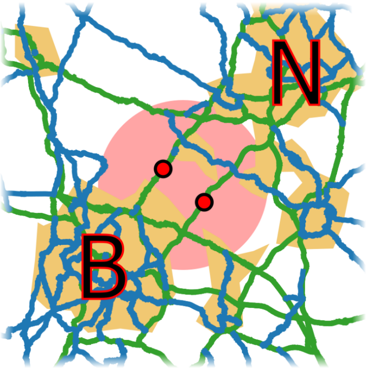
- $X \subseteq V(G)$ is *separator* if $G - X$ has $\geqslant 2$ components
- *separation* of G: pair (A, B) with $A \cup B = V(G)$ and $A \setminus B \neq \emptyset \neq B \setminus A$ but without $(A \setminus B)$–$(B \setminus A)$ edges
- $A \cap B$ is the *separator* of (A, B)

- $(A, B) \leqslant (C, D) :\Leftrightarrow A \subseteq C$ and $B \supseteq D$
- (A, B) and (C, D) are *nested* if $(A, B) \leqslant (C, D)$ possibly after switching roles of A, B or of C, D; otherwise they *cross*
- a set of separations is *nested* if its elements are pairwise nested

Recent development in Graph Minor Theory:

local separators





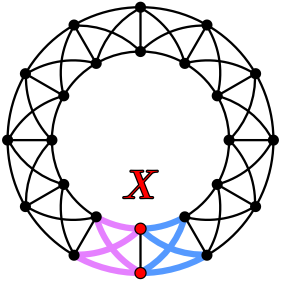
Rough idea: vertex-sets that separate G locally in a ball of given radius, not necessarily G itself

Given G, $r > 0$ and $X \subseteq V(G)$.

Two edges $e, f \in \partial X$ *lie in the same r-local component* at X if

- there are a cycle $O \subseteq G$ of length $\leqslant r$, and
- a subpath P of O that starts with e and ends with f,
- such that P only meets X in its endvertices.

We allow $P = O$.



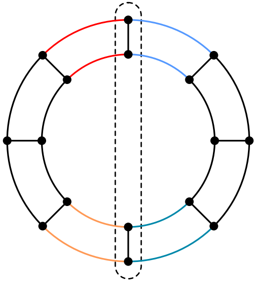
$3 \leqslant r < 12$

Two vertices of X *lie in the same r-local atom* of X if

- they lie together on a cycle of length $\leqslant r$, or
- are joined by an edge.

Call X *r-locally atomic*, or *r-tomic*, if X consists of one r-tom.

Example of <u>not</u> r-tomic:

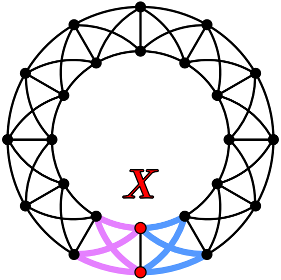


X is an *r-local separator* if

- there are least two r-local components at X, and
- X is r-tomic.

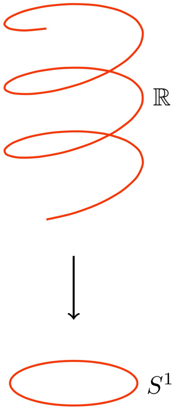
An *r-local separation* is a triple (E, X, F) where

- X is an r-local separator, and
- E, F bipartition ∂X while respecting r-local components.



A *covering* of G is a surjective graph-homomorphism $p\colon C \to G$ such that for every vertex $v \in C$:

- p restricts to a bijection $\partial_C(v) \to \partial_G(p(v))$.

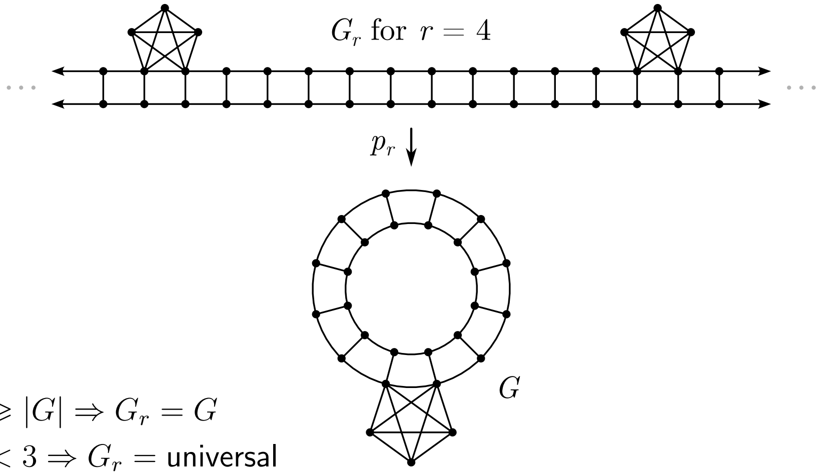


Example: universal coverings are trees.

The *ball $B_G(v, r/2)$ of radius $r/2$* around $v \in V(G)$ consists of all vertices and edges that lie on closed walks of length $\leqslant r$ through v.

$\forall G$ and $r > 0$ there is a unique *r-local covering $p_r \colon G_r \to G$* s.t.

1. p_r restricts to an isomorphism $B_{G_r}(v, r/2) \to B_G(p_r(v), r/2)$ for every $v \in V(G_r)$, and
2. p_r is 'nearest' to the universal covering with (1).



$r \geqslant |G| \Rightarrow G_r = G$

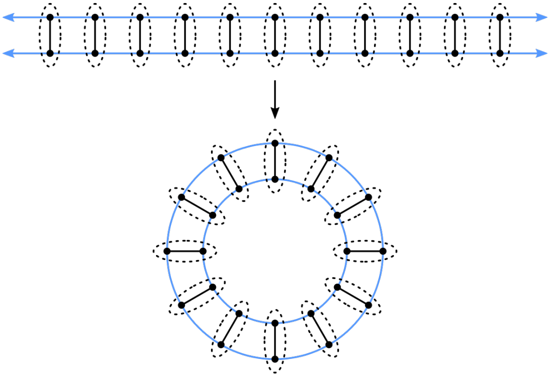
$r < 3 \Rightarrow G_r =$ universal

separations of $G_r \hat{=}$ lifts of r-local separations of G (roughly)

Two r-local separations of G are *nested* if all their lifts are nested.

Ideas @challenges for finite Γ:

1. Use ends of r-local covering of some $\mathrm{Cay}(\Gamma, S)$.
2. Use Γ-orbit of suitable r-local separation in proof.



Main result (Carmesin, Kontogeorgiou, K., Turner '24)

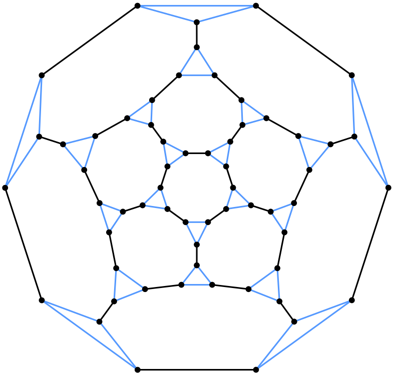
Let Γ be a finite group that is nilpotent of class $\leqslant n$.
Let $r \geqslant \max\{4^{n+1}, 20\}$. Then TFAE:

1. The r-local covering of some Cayley graph G of Γ has $\geqslant 2$ ends that are separated by $\leqslant 2$ vertices.
2. G has an r-local separator of size $\leqslant 2$ and $|\Gamma| > r$.
3. $\Gamma \cong C_i \times C_j$ for some $i > r$ and $j \in \{1, 2\}$.

Questions:

- Why nilpotent?
- Why only (local) separators of size $\leqslant 2$?

Why nilpotent?



(1) and (2) hold for $r \leqslant 9$.

(3) cannot be amended (A_5 is simple).

Open problem (in reach): Extend main result to solvable groups.

Why only (local) separators of size $\leqslant 2$?

Heavily exploited in proof...

G has r-local separator X with $|X| \leqslant 2$ and $|\Gamma| > r$

$\Rightarrow$ $\Gamma \cong C_i \times C_j$ for some $i > r$ and $j \in \{1, 2\}$.

Proof. Say $G = \text{Cay}(\Gamma, S)$.

Case $|X| = 1$: We claim $S = \{s^{\pm 1}\}$ (so G is a cycle).

Suppose for a contradiction that $\{s^{\pm 1}\} \subsetneq S$.

It suffices to show that $B(\mathbb{I}, r/2) - \mathbb{I}$ is connected.

We show that every $g^{\pm 1} \neq h^{\pm 1} \in S$ lie in the same component.

$[g, h]_n = \mathbb{I} \Longrightarrow$ short closed walk $\Longrightarrow$ short cycle O.
($[g, h]_1 := gh^{-1}g^{-1}h$ and $[g, h]_n := [g, [g, h]_{n-1}]_1$ after reduction)

At least three of the words $gh, g^{-1}h, gh^{-1}, hg$ occur on O
(we may use both directions of O to find words). $\diamondsuit$

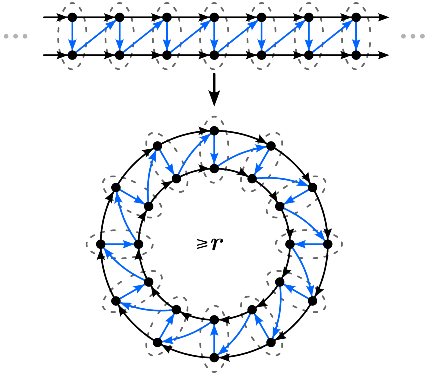
G has r-local separator X with $|X| \leqslant 2$ and $|\Gamma| > r$

$\Rightarrow$ $\Gamma \cong C_i \times C_j$ for some $i > r$ and $j \in \{1, 2\}$.

Proof (continued). Case $|X| = 2$: WLOG $X = \{\mathbb{I}, h\}$.

Assume for now that $h \in S$.

Subcase $h^2 \neq \mathbb{I}$: We show $S \subseteq \{h^{\pm 1}, h^{\pm 2}\}$.



G has r-local separator X with $|X| \leqslant 2$ and $|\Gamma| > r$

$\Rightarrow$ $\Gamma \cong C_i \times C_j$ for some $i > r$ and $j \in \{1, 2\}$.

Proof (continued). Case $|X| = 2$: WLOG $X = \{\mathbb{I}, h\}$.

Assume for now that $h \in S$.

Subcase $h^2 = \mathbb{I}$: We show that $\langle h \rangle$ is a normal subgroup of Γ.

Obtain G' from G by contracting all h-labelled edges.

G' is a Cayley graph of $\Gamma/\langle h \rangle$ with local cutvertex X.

So G' and $\Gamma/\langle h \rangle$ are cyclic by Case $|X| = 1$.

Thus $\Gamma \cong C_i \times C_2$ with $i > r$. $\diamondsuit$

G has r-local separator X with $|X| \leqslant 2$ and $|\Gamma| > r$

$\Rightarrow$ $\Gamma \cong C_i \times C_j$ for some $i > r$ and $j \in \{1, 2\}$.

Proof (continued). <u>Case $|X| = 2$:</u> WLOG $X = \{\mathbb{I}, h\}$.

Assume for now that $h \in S$.

We cannot be greedy and add h to S.

Theorem (Tutte 60s): Every 2-connected graph is

- 3-connected,
- has a 2-separation that is nested with all 2-separations, or
- is a cycle.

G has r-local separator X with $|X| \leqslant 2$ and $|\Gamma| > r$

$\Rightarrow$ $\Gamma \cong C_i \times C_j$ for some $i > r$ and $j \in \{1, 2\}$.

Proof (continued). Case $|X| = 2$: WLOG $X = \{\mathbb{I}, h\}$.

Assume for now that $h \in S$.

We cannot be greedy and add h to S.

Theorem (Carmesin '20): Every r-**locally** 2-connected graph is

- r-**locally** 3-connected,
- has an r-**local** 2-separation that is nested with all r-**local** 2-separations, or
- is a cycle **of length** $\leqslant r$.

Choose X nested with all r-local 2-separations, **then** add h to S.

□

Outlook

Open problem: Extension to solvable groups (and beyond).

Open problem: Extension to (local) separators of size > 2.

Big question: What types of products will occur?

Main result. Let Γ be a finite group that is nilpotent of class $\leqslant n$. Let $r \geqslant \max\{4^{n+1}, 20\}$. Then TFAE:

1. The r-local covering of some Cayley graph G of Γ has $\geqslant 2$ ends that are separated by $\leqslant 2$ vertices.
2. G has an r-local separator of size $\leqslant 2$ and $|\Gamma| > r$.
3. $\Gamma \cong C_i \times C_j$ for some $i > r$ and $j \in \{1, 2\}$.

Open: (1) Solvable groups. (2) Large local separators.

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Thank you!