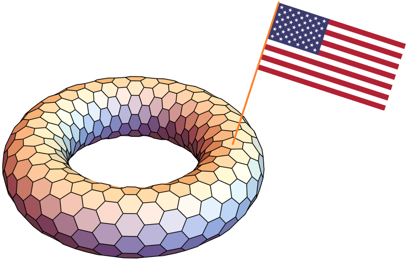
Canonical decompositions of 3-connected graphs

Jan Kurkofka



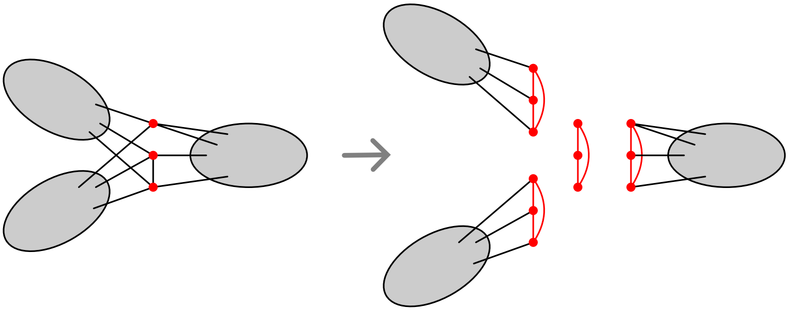
Joint work with Johannes Carmesin

University of Birmingham

FOCS 2023

Problem: Decompose k-con'd G along k-separators into pieces that are $(k+1)$-con'd or 'basic'.

Decomposing G along a k-separator:



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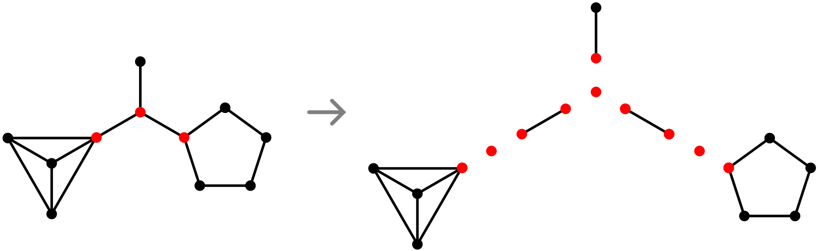
$k = 1$:



Problem: Decompose k-con'd G along k-separators
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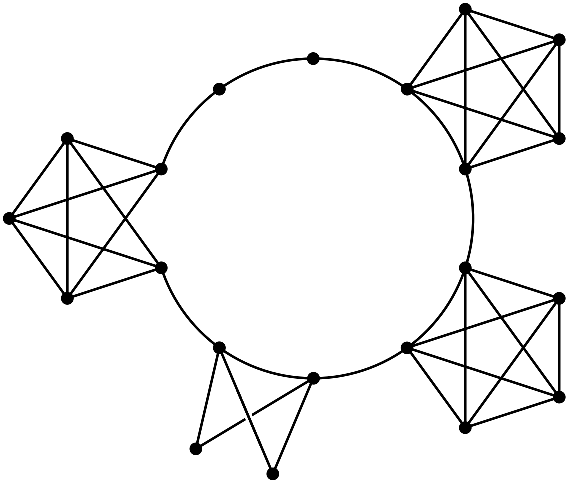
$k = 2$:



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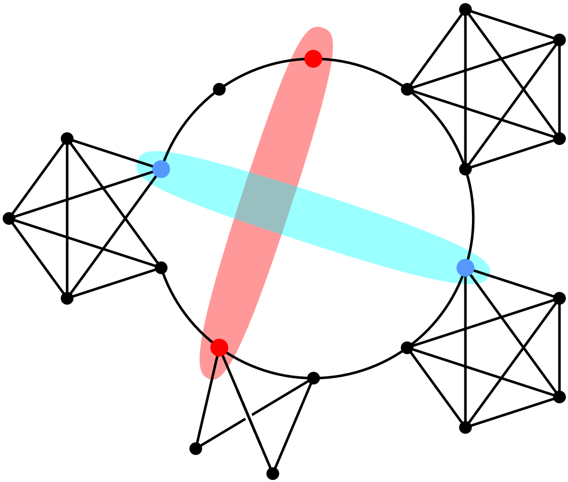

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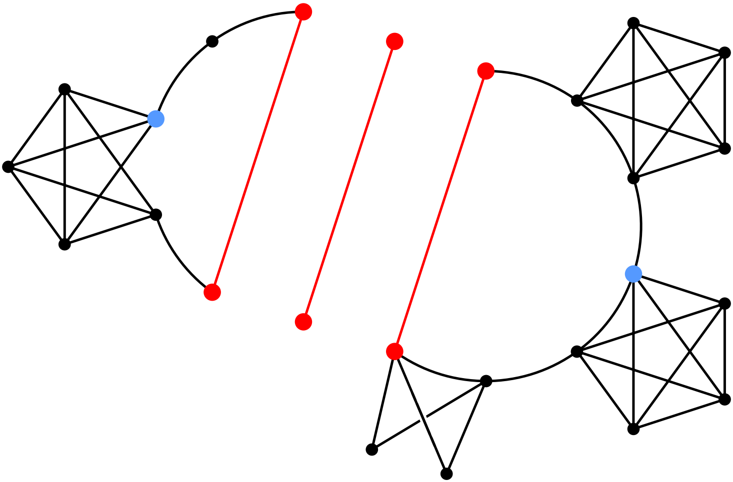
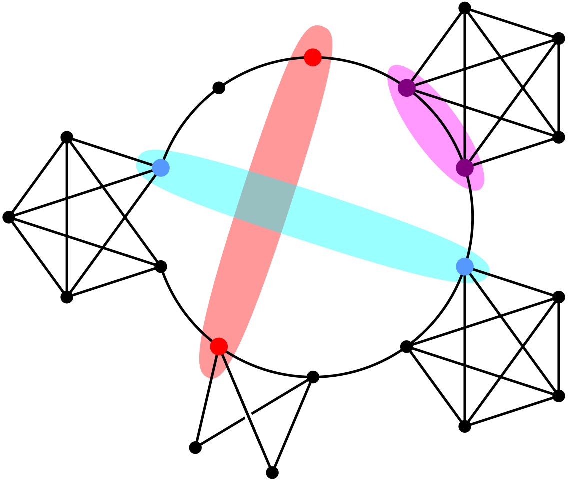


Two k-separators *cross* if they separate each other;
otherwise they are *nested*.



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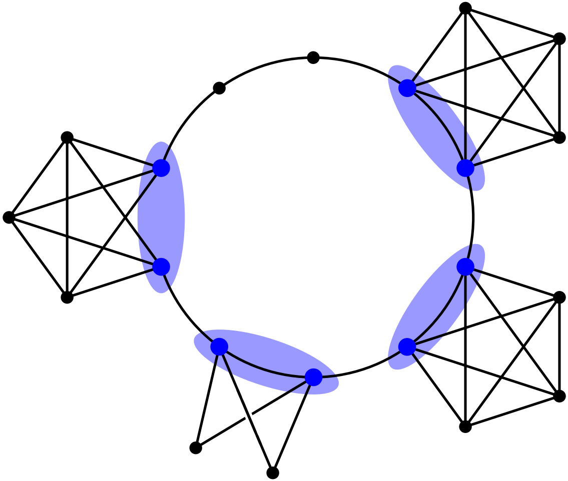

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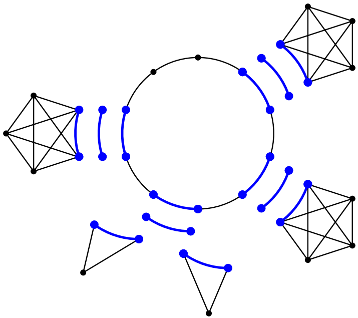
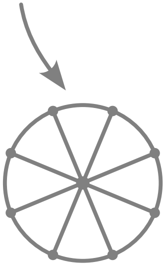


Theorem (Tutte 66), SPQR-trees

Every 2-con'd G decomposes along its totally-nested 2-separators into 3-con'd graphs, cycles and K_2's.

Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into 4-con'd graphs, wheels and K_3's.



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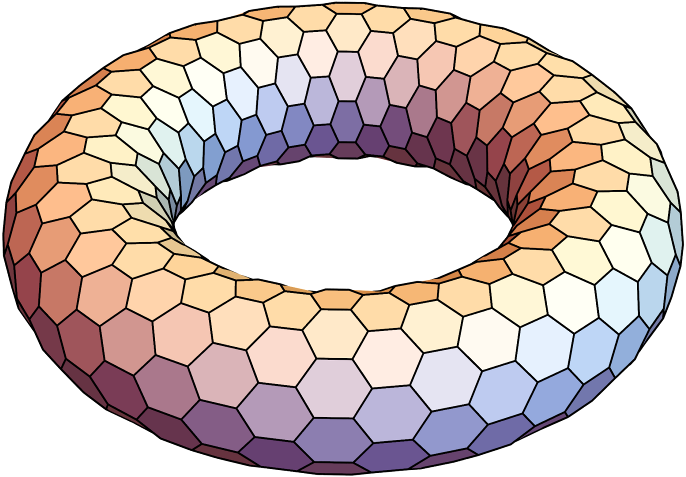

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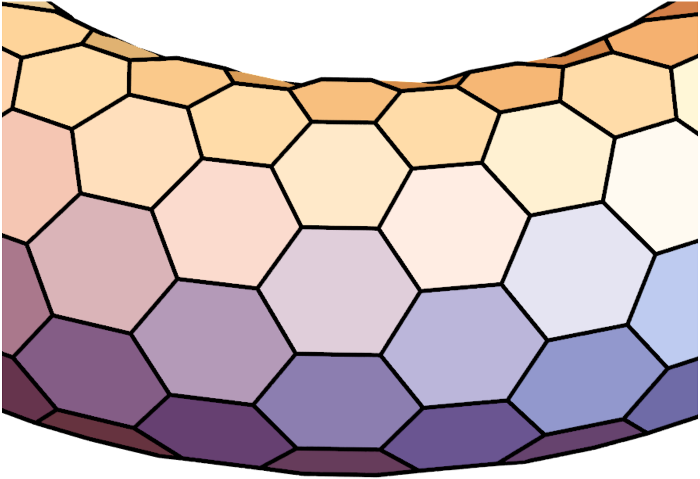

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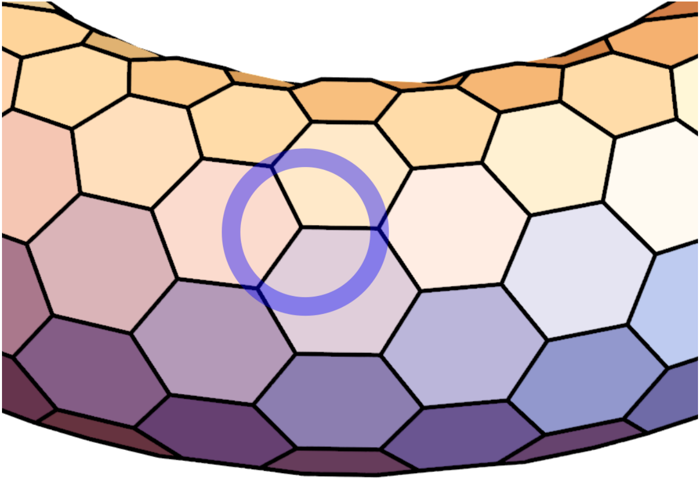

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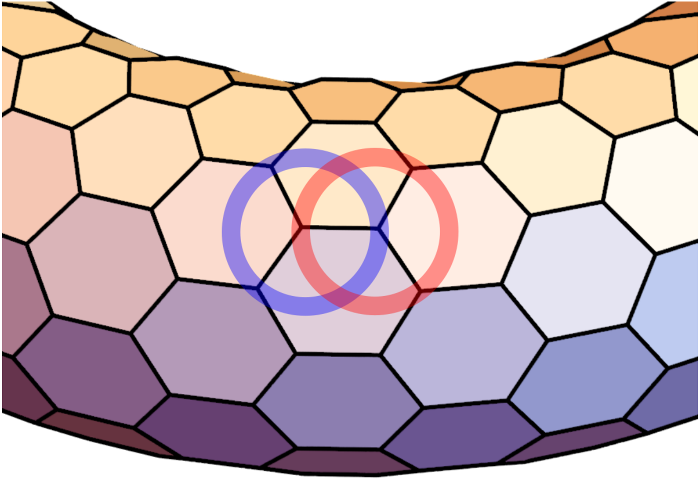
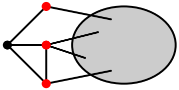
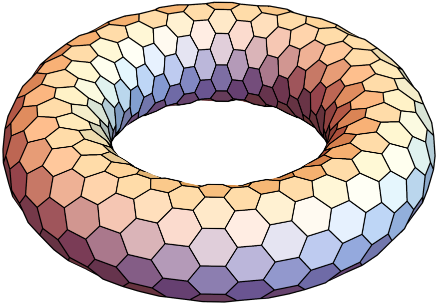


Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into quasi 4-con'd graphs, wheels and K_3's.

3-con'd, >4 vertices, every 3-separator has form 



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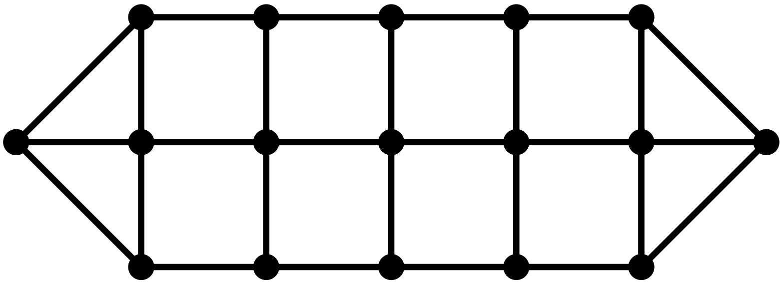

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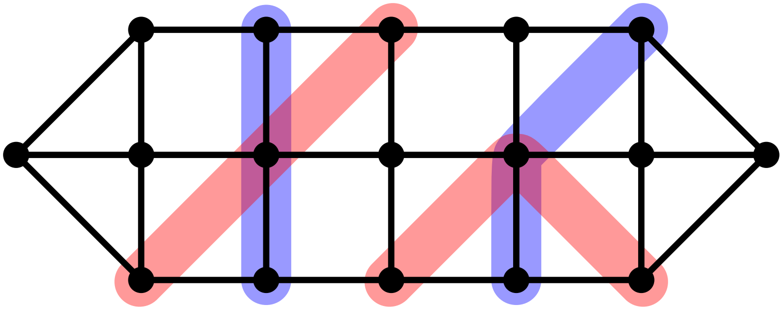

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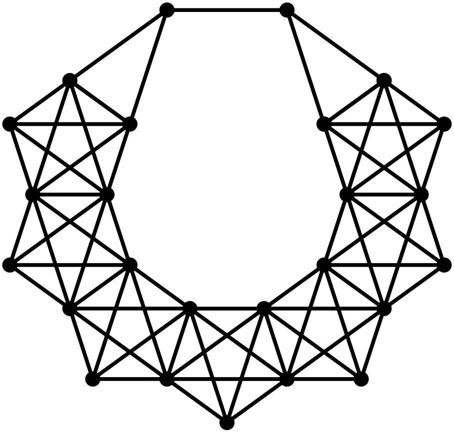

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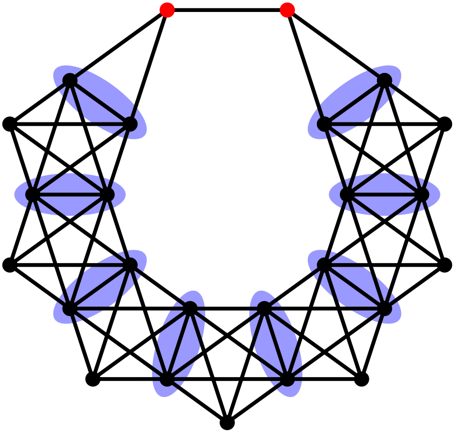

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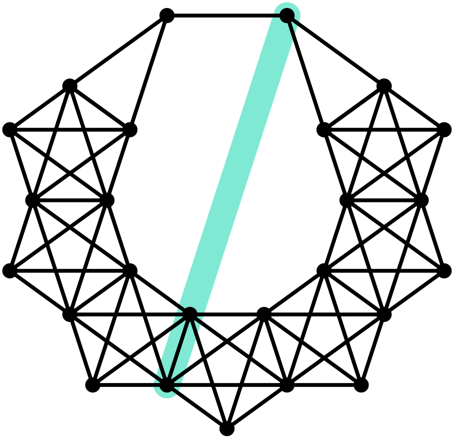

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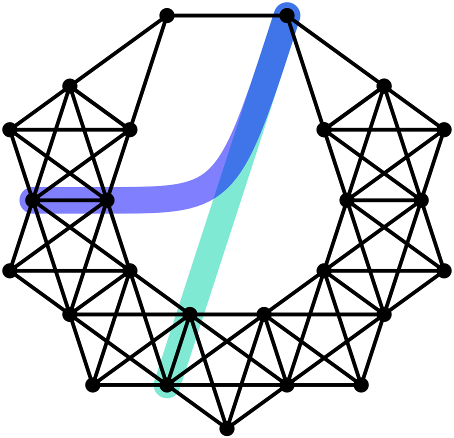

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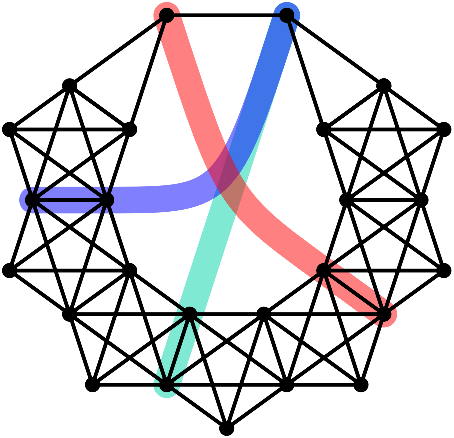

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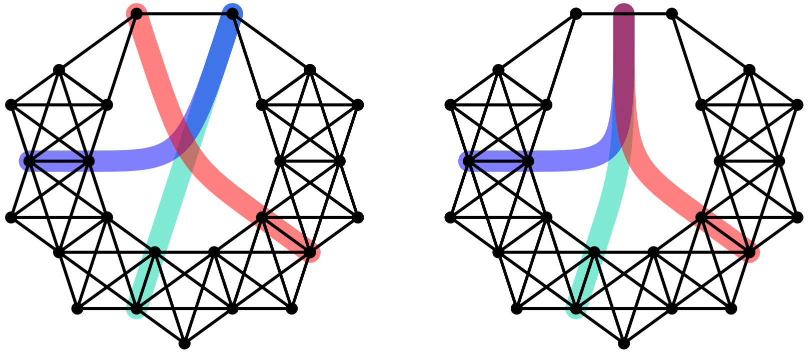

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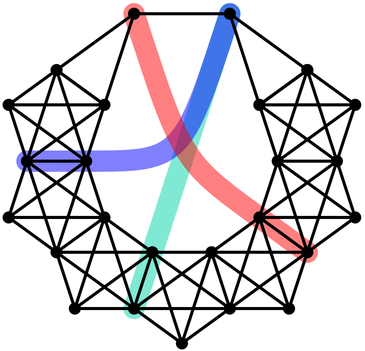

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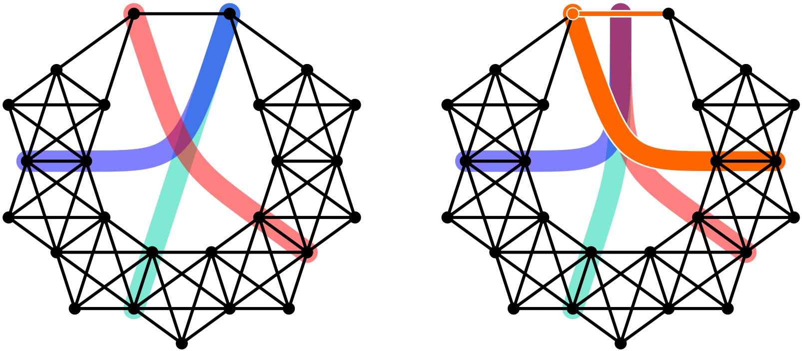

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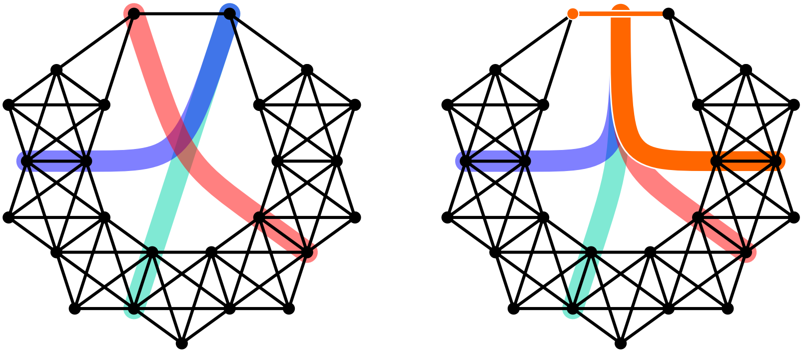
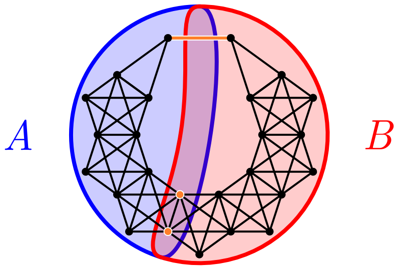
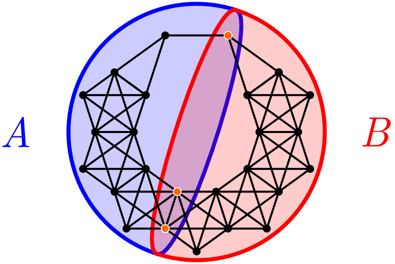




mixed-separation of G:	$\{A, B\}$ with $A \cup B = V(G)$ and both $A \setminus B$ and $B \setminus A$ nonempty
separator of $\{A, B\}$:	$(A \cap B) \cup E(A \setminus B, B \setminus A)$
tri-separation of G:	mixed-sep'n $\{A, B\}$ with $\lvert\text{sep'r}\rvert = 3$ and every vx in $A \cap B$ has two neighb's in $G[A]$ and in $G[B]$



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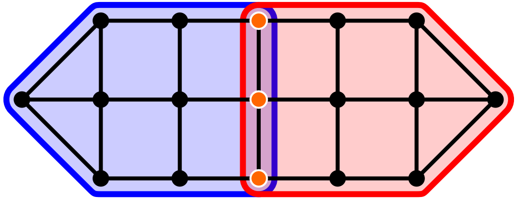

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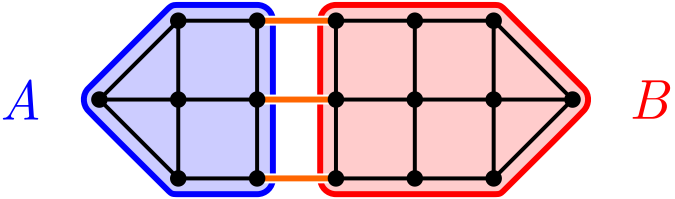

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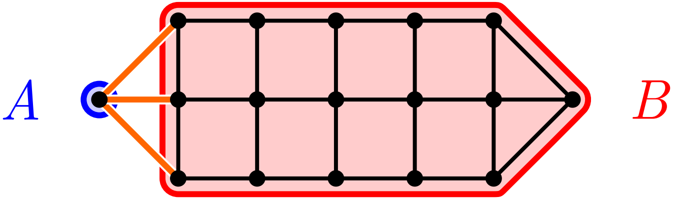
trivial



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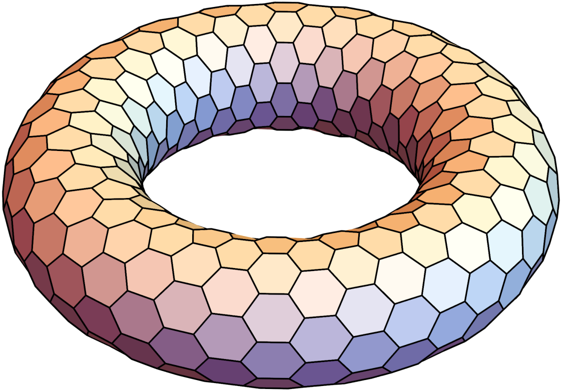
totally-nested nontrivial tri-separations



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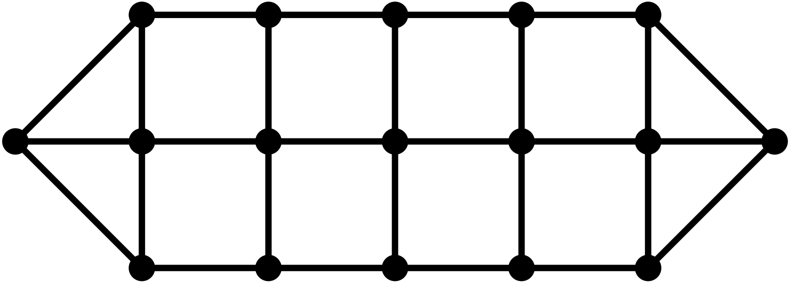

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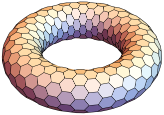

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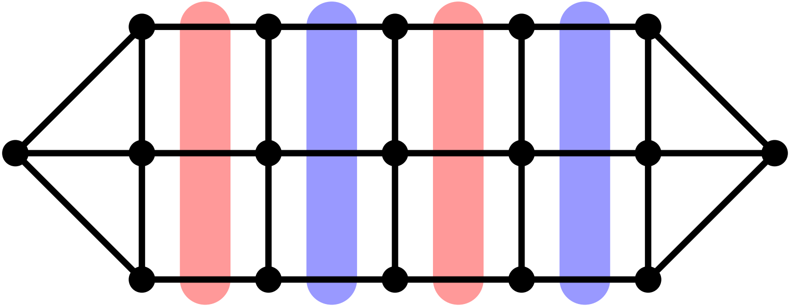

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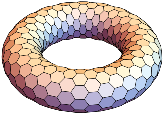

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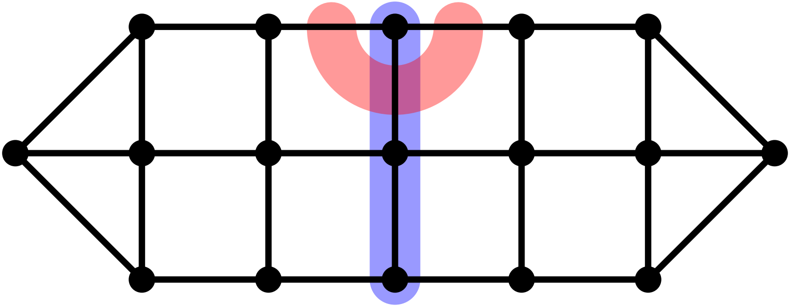

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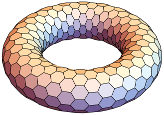

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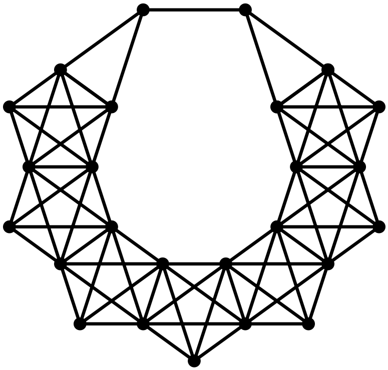

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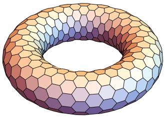

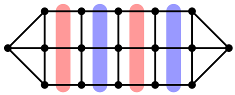

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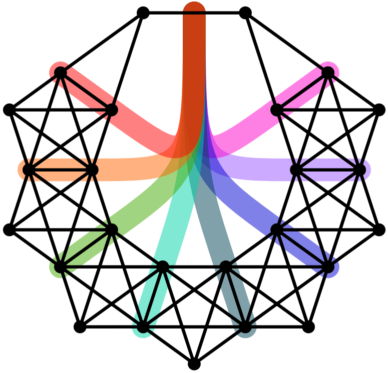

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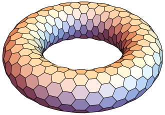
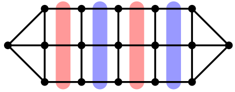
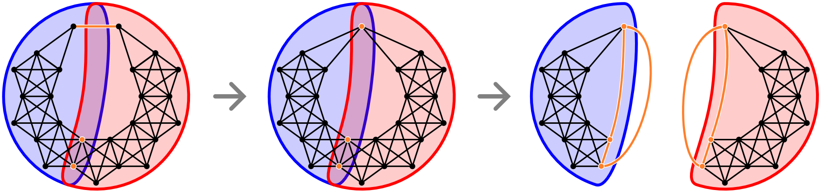
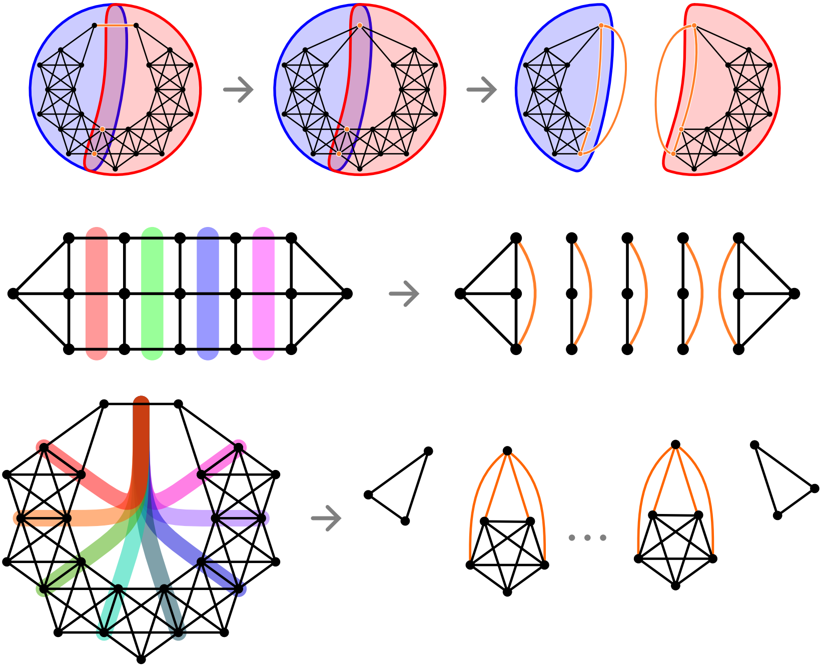




Decomposing along a tri-separation



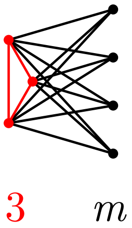


Main result (Carmesin & K. 23)

Every 3-con'd G decomposes along its totally-nested nontrivial tri-separations into minors of G that are

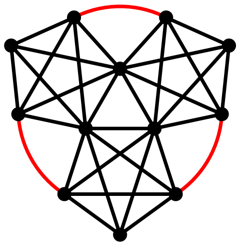
- quasi 4-con'd
- wheels
- thickened $K_{3,m}$

or $G = K_{3,m}$ $(m \geqslant 0)$.



	Grohe 16	Carmesin & K. 23
method	recursive	Tutte (totally nested)
decomposition	3-separations	tri-separations
torsos	K_4,	wheels,
	quasi 4-con'd, K_3	quasi 4-con'd, thickened $K_{3,m}$
canonical	no	yes
algorithm	$O(n^2(n+m))$	???

Application: Connectivity Augmentation from 0 to 4



Theorem (Carmesin & Sridharan 23+)

$\exists$ FPT-algorithm with runtime $C(\ell) \cdot \mathrm{Poly}(\,|V(G)|\,)$ and

Input: Graph G, $\ell \in \mathbb{N}$ and $F \subseteq E(\,\overline{G}\,)$

Output: No, or $\leqslant \ell$-sized $X \subseteq F$ such that $G + X$ is 4-con'd

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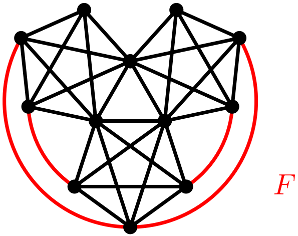

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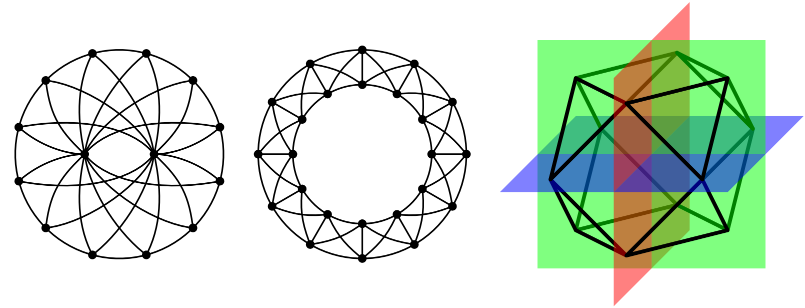
Open: Extend the main result to k-separations for $k \geqslant 4$.



Open: Efficient algorithms?

Open: Directed graphs?

$k = 1$: Bowler, Gut, Hatzel, Kawarabayashi, Muzi, Reich 23

$k \geqslant 2$: ???

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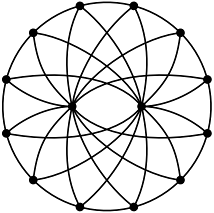
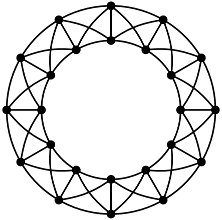
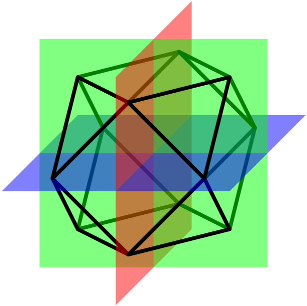

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Thank you!