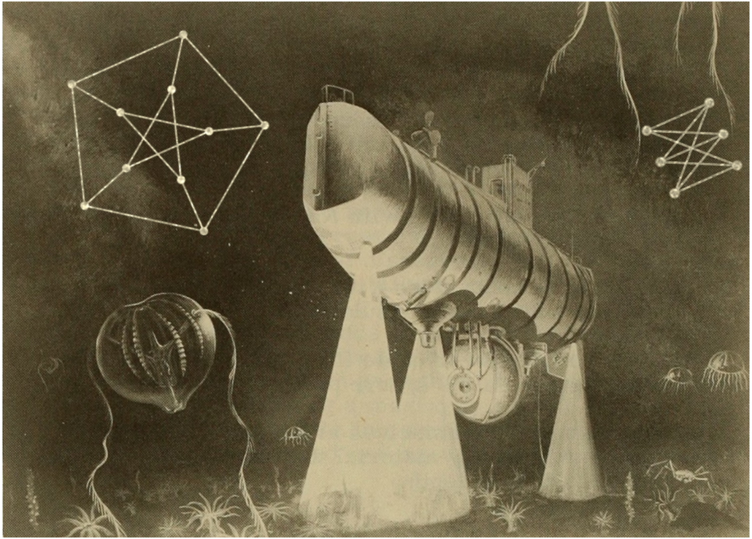
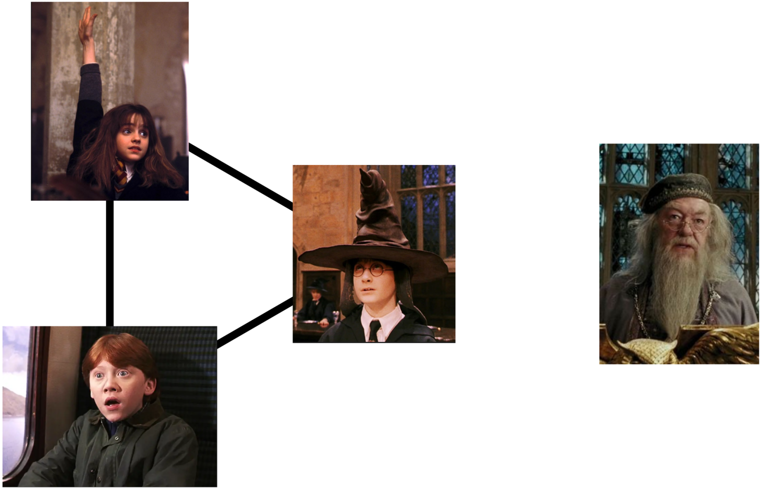
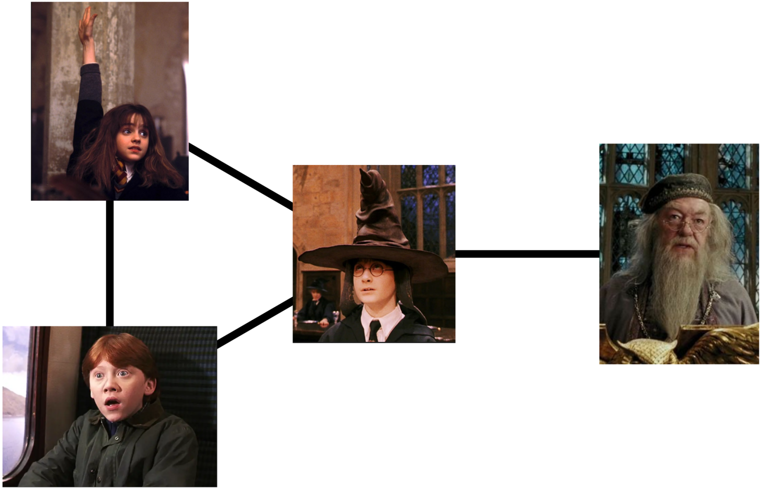
A Combinatorial Journey to the Challenger Deep of Mathematics

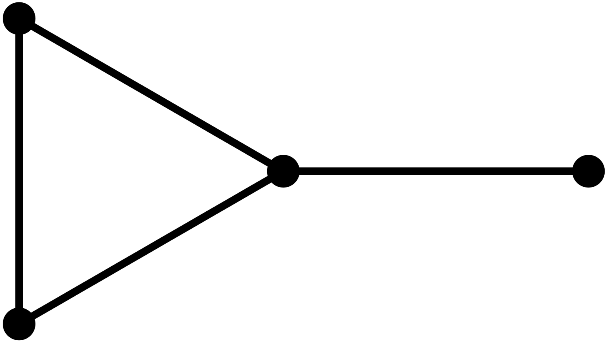
Jan Kurkofka



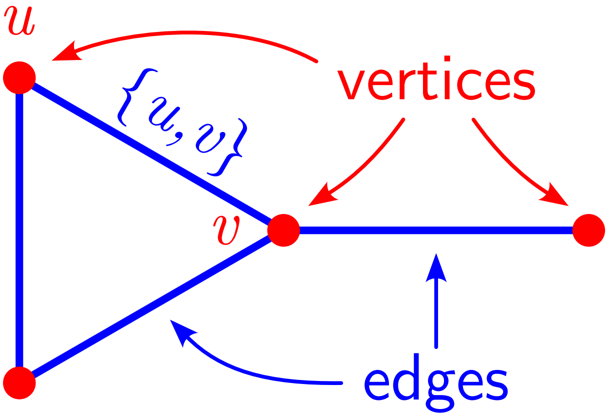
Social networks



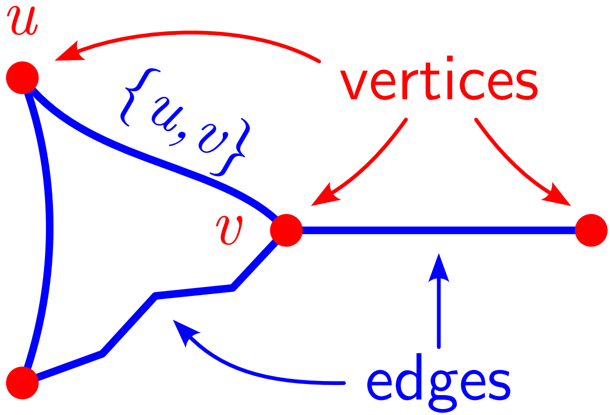




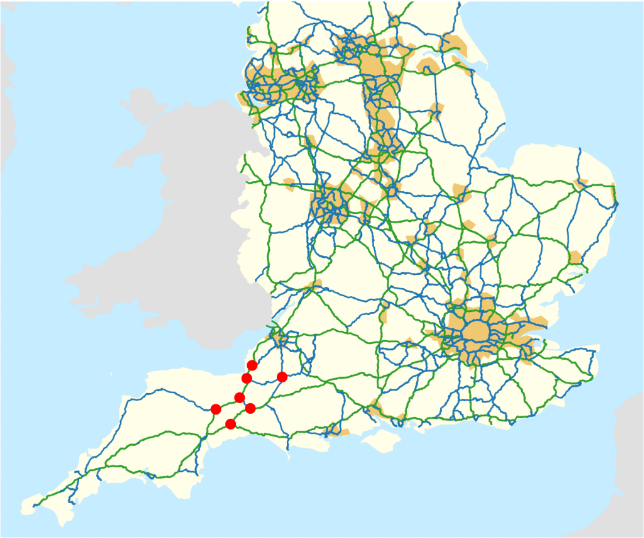
graph



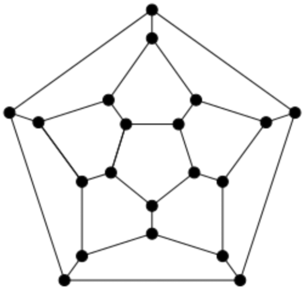
graph



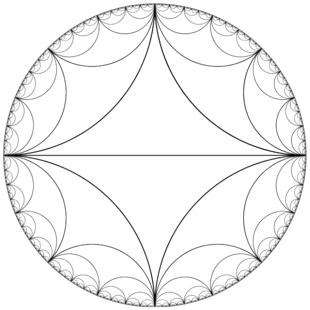
Infrastructure



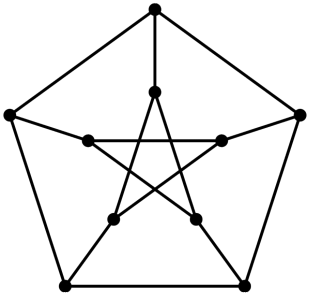
Maths!



Dodecahedron



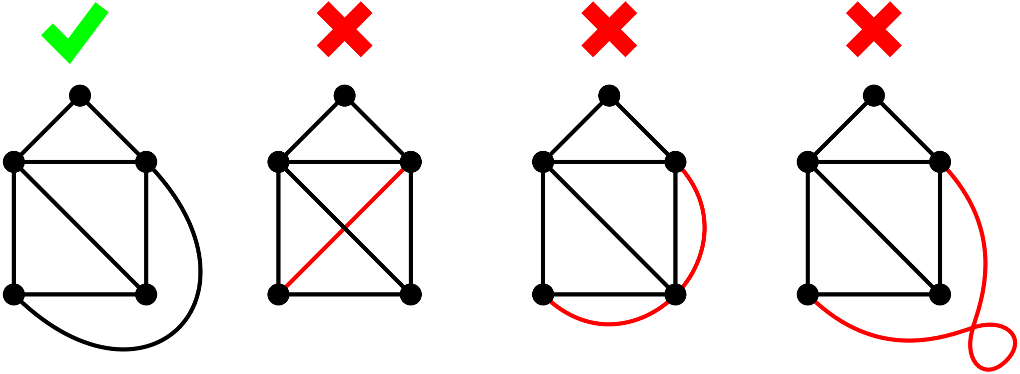
Farey graph



Petersen graph

Which graphs can be drawn in the plane so that no two edges cross?

are planar

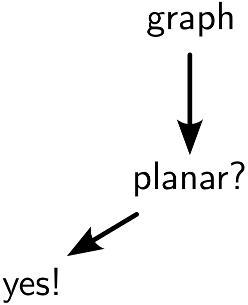


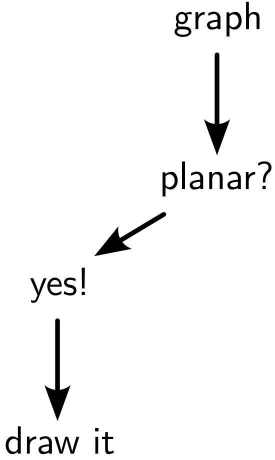
graph

graph

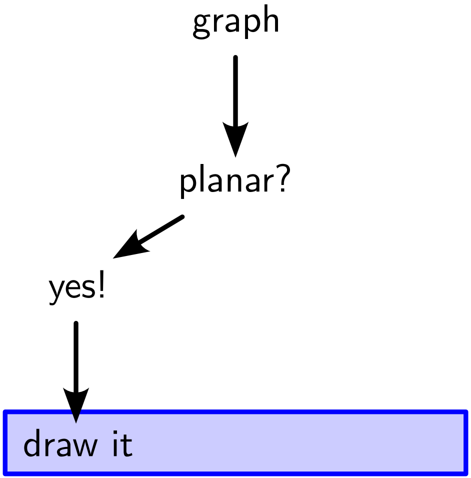
↓

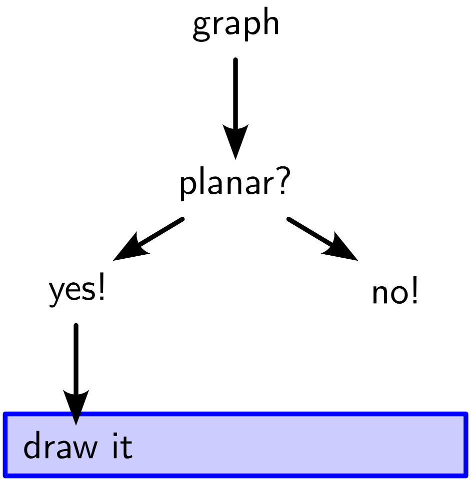
planar?


graph
planar?
yes!


graph
planar?
yes!
draw it

Easy to verify





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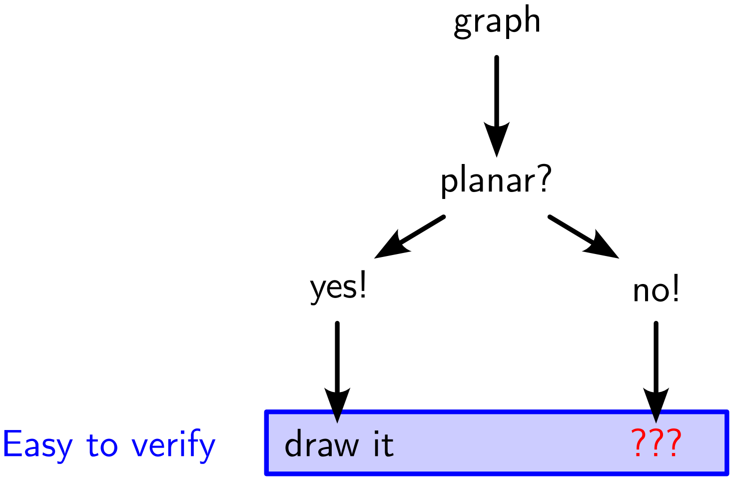

graph
planar?
yes!
no!
Easy to verify
draw it
???

Find a nonplanar graph

Find a nonplanar graph

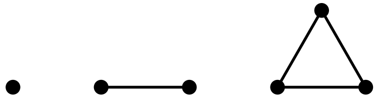
K_1

Find a nonplanar graph

K_1

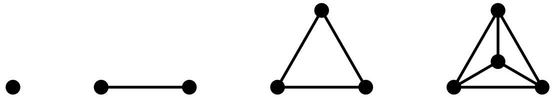
K_2

Find a nonplanar graph



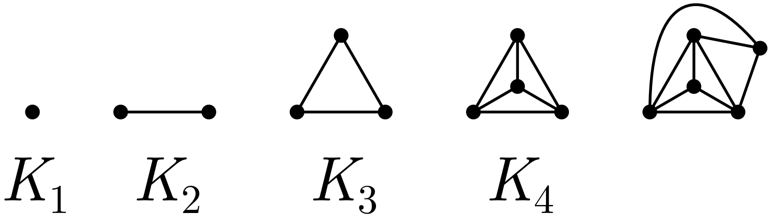
K_1 K_2 K_3

Find a nonplanar graph

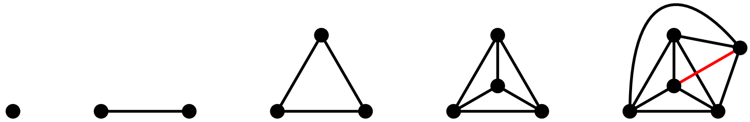


K_1 K_2 K_3 K_4

Find a nonplanar graph



Find a nonplanar graph



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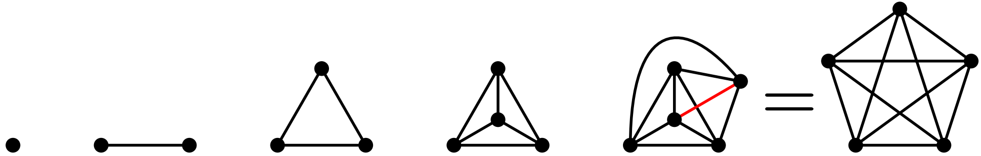
Find a nonplanar graph



K_1 K_2 K_3 K_4 K_5

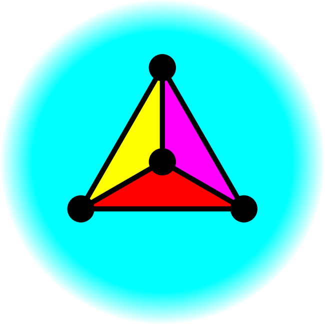
Euler's formula (1752)

For every planar drawing of a connected graph with n vertices and m edges:

$$n - m + \ell = 2$$

where ℓ is the number of faces of the drawing.

faces: connected regions of the plane minus the drawing



$$\begin{aligned} n &= 4 \\ m &= 6 \\ \ell &= 4 \end{aligned}$$

Euler's formula (1752)

For every planar drawing of a connected graph with n vertices and m edges:

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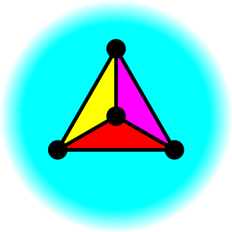

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Corollary A. Every triangulation of the plane with n vertices has $3n - 6$ edges.

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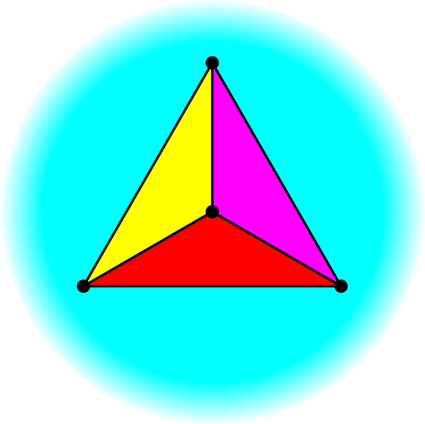

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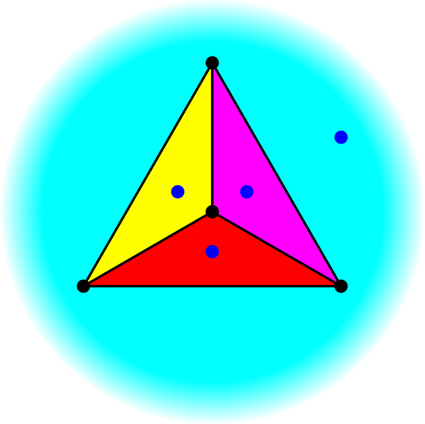

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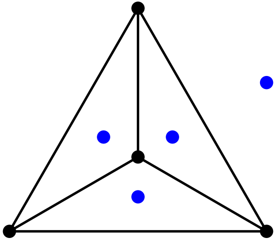

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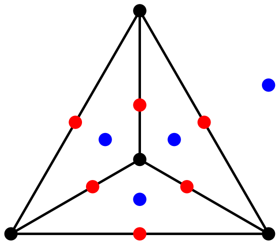

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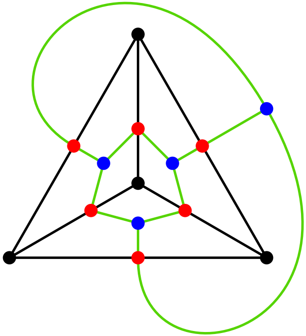

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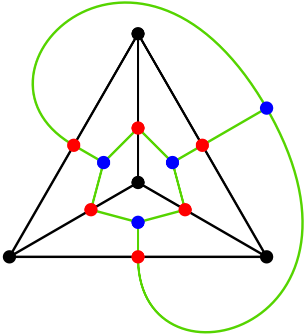

#green edges

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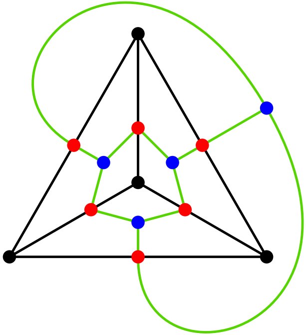

two perspectives

from faces

from edges

#green edges

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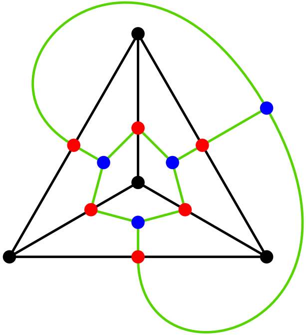

two perspectives

from faces

from edges

$3\ell =$ #green edges

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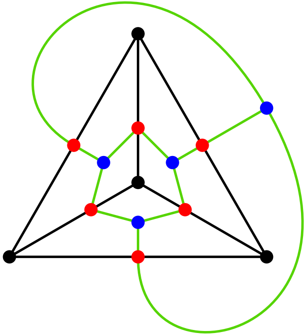

two perspectives

from faces | | from edges

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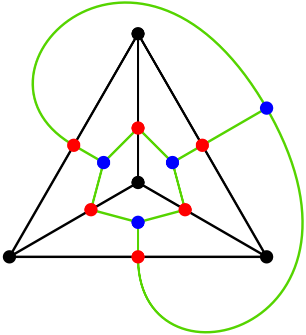

two perspectives

from faces | | from edges
$3\ell =$ | #green edges | $= 2m$

$$\Longrightarrow \ell = \frac{2}{3}m \quad (*)$$

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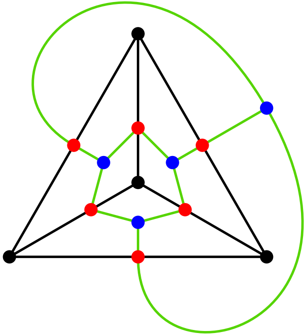

$$(*) \qquad \ell = \tfrac{2}{3}m$$

Euler's formula: $$n - m + \ell = 2$$

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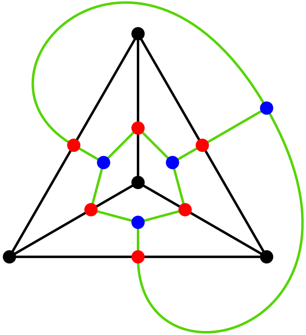

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$$\overset{(*)}{\Longrightarrow} \quad n - m + \tfrac{2}{3}m = 2$$

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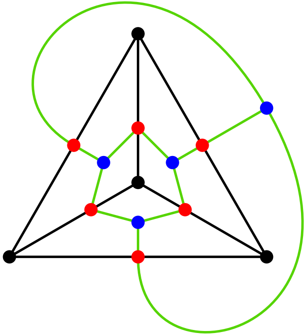

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$$\overset{(*)}{\Longrightarrow} \quad n - m + \frac{2}{3}m = 2$$

$$\Longrightarrow \quad n - \frac{1}{3}m = 2$$

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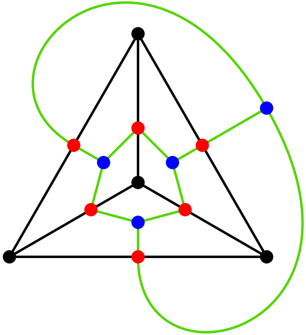

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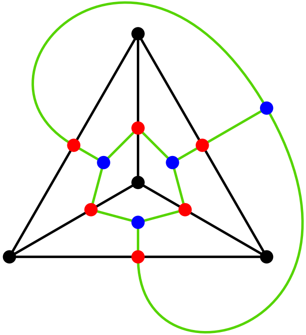
Euler's formula:
$$\begin{aligned} n - m + \ell &= 2 \\ \overset{(*)}{\Longrightarrow} \quad n - m + \tfrac{2}{3}m &= 2 \\ \Longrightarrow \quad n - \tfrac{1}{3}m &= 2 \\ \Longrightarrow \quad n - 2 &= \tfrac{1}{3}m \end{aligned}$$

Corollary A. Every triangulation of the plane with n vertices has $3n - 6$ edges.

Proof.



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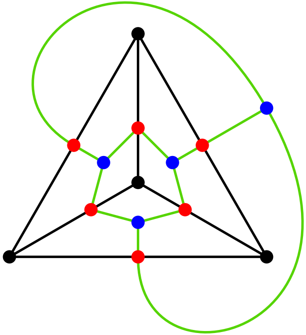

$$(*) \qquad \ell = \tfrac{2}{3}m$$

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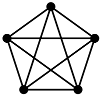
$$\begin{aligned} n - m + \ell &= 2 \\ \overset{(*)}{\Longrightarrow} \quad n - m + \tfrac{2}{3}m &= 2 \\ \Longrightarrow \quad n - \tfrac{1}{3}m &= 2 \\ \Longrightarrow \quad n - 2 &= \tfrac{1}{3}m \\ \Longrightarrow \quad 3n - 6 &= m \end{aligned}$$

□

Corollary A. Every triangulation of the plane with n vertices has $3n - 6$ edges.

Corollary B. K_5  is not planar.

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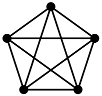
Corollary B. K_5  is not planar.

Proof. Assume for a contradiction that K_5 is planar.

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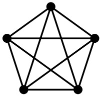
Corollary B. K_5  is not planar.

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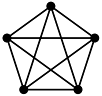
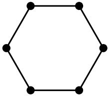
Draw it!

Corollary A. Every triangulation of the plane with n vertices has $3n-6$ edges.

Corollary B. K_5  is not planar.

Proof. Assume for a contradiction that K_5 is planar.

Draw it! Without proof: every face is bounded by a cycle 

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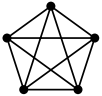
Corollary B. K_5  is not planar.

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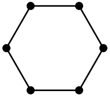

This is a triangulation:

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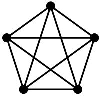
Corollary B. K_5  is not planar.

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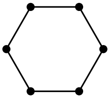
Draw it! Without proof: every face is bounded by a cycle 

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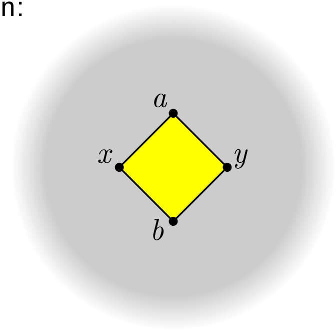

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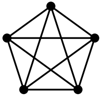
Corollary B. K_5  is not planar.

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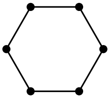
Draw it! Without proof: every face is bounded by a cycle 

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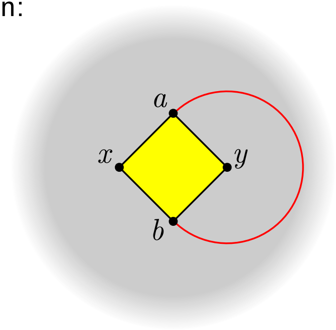
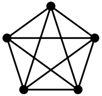


Corollary A. Every triangulation of the plane with n vertices has $3n - 6$ edges.

Corollary B. K_5  is not planar.

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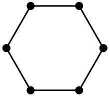
Draw it! Without proof: every face is bounded by a cycle 

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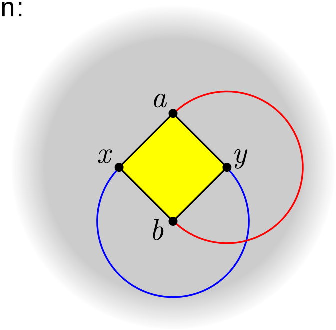

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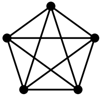
Corollary B. K_5  is not planar.

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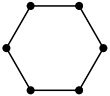
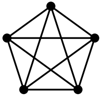


This is a triangulation.

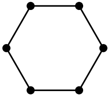
Corollary A says K_5 has $3n-6=3\cdot 5-6=9$ edges.

Corollary A. Every triangulation of the plane with n vertices has $3n-6$ edges.

Corollary B. K_5  is not planar.

Proof. Assume for a contradiction that K_5 is planar.

Draw it! Without proof: every face is bounded by a cycle

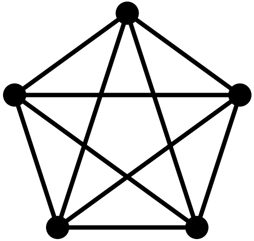


This is a triangulation.

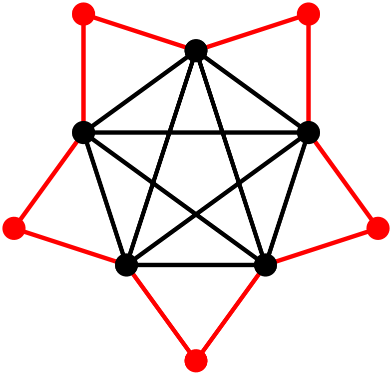
Corollary A says K_5 has $3n-6=3\cdot 5-6=9$ edges.

But K_5 has $\binom{5}{2}=10$ edges, contradiction. $\square$

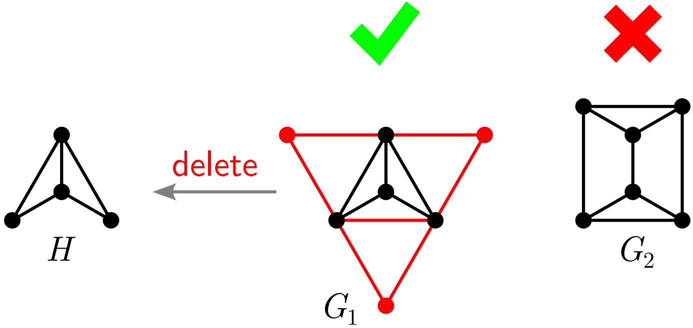
K_5 is not planar. Are there other nonplanar graphs?



K_5 is not planar. Are there other nonplanar graphs? Yes!

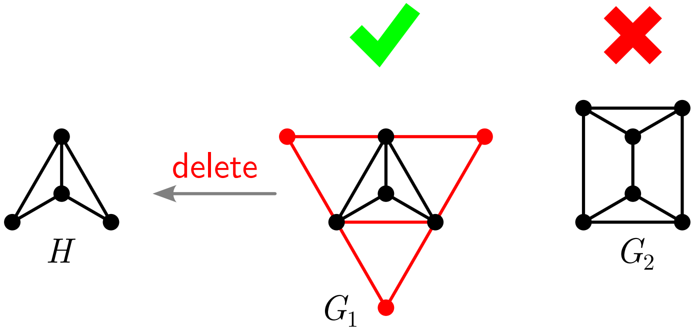


A graph H is a *subgraph* of a graph G if H can be obtained from G by successively deleting edges or isolated vertices.



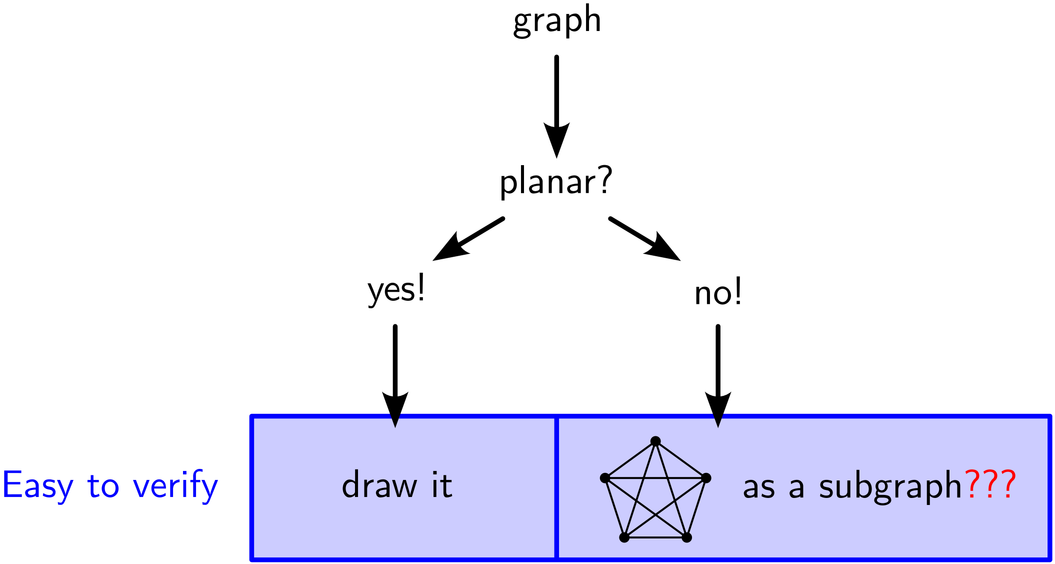
Fact. Subgraphs of planar graphs are planar.

A graph H is a *subgraph* of a graph G if H can be obtained from G by successively deleting edges or isolated vertices.



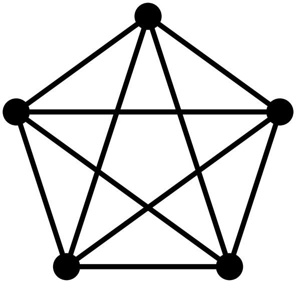
Fact. Subgraphs of planar graphs are planar.

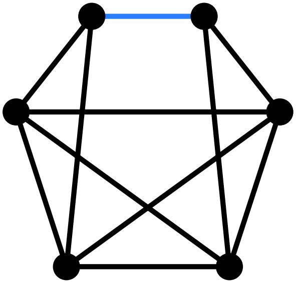
Conjecture. Every nonplanar graph contains K_5 as a subgraph.


graph
planar?
yes!
no!
Easy to verify
draw it
as a subgraph???

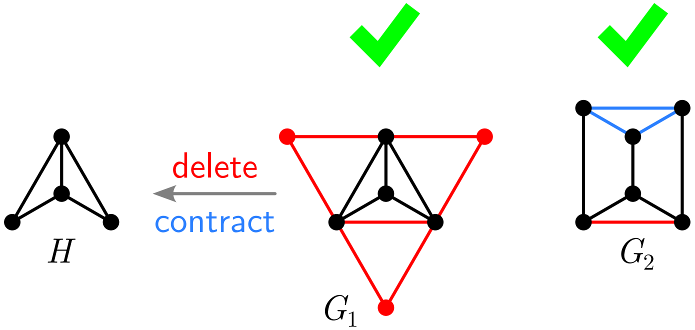
Is the right graph planar?

Does the right graph contain K_5 as a subgraph?





A graph H is a *minor* of a graph G if H can be obtained from G by successively deleting edges or isolated vertices or contracting edges.

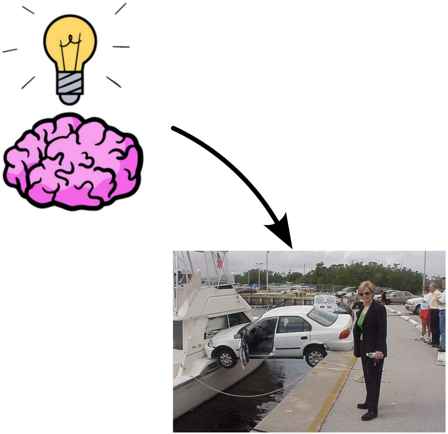


Fact. Minors of planar graphs are planar.

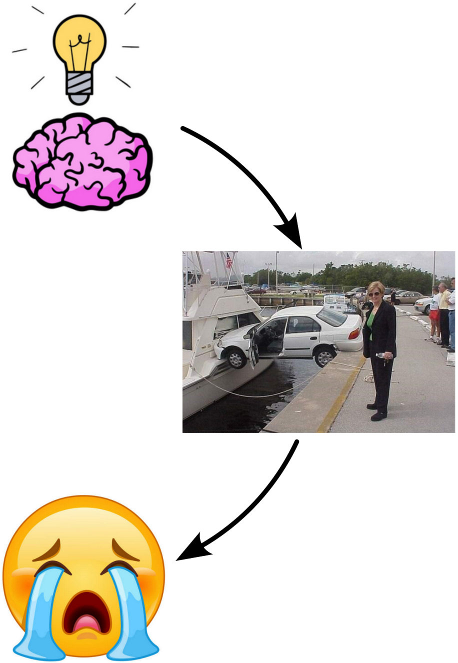
Conjecture. Every nonplanar graph contains K_5 as a minor.

Life as a Mathematician

Life as a Mathematician



Life as a Mathematician



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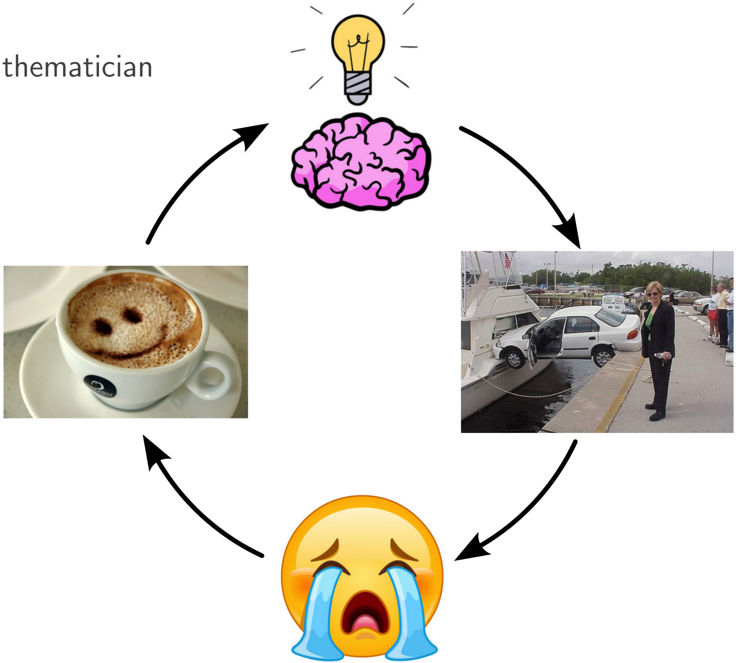
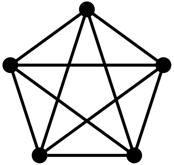


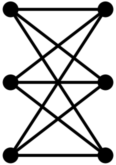
Kuratowski's theorem (1930)

For every graph G, the following assertions are equivalent:

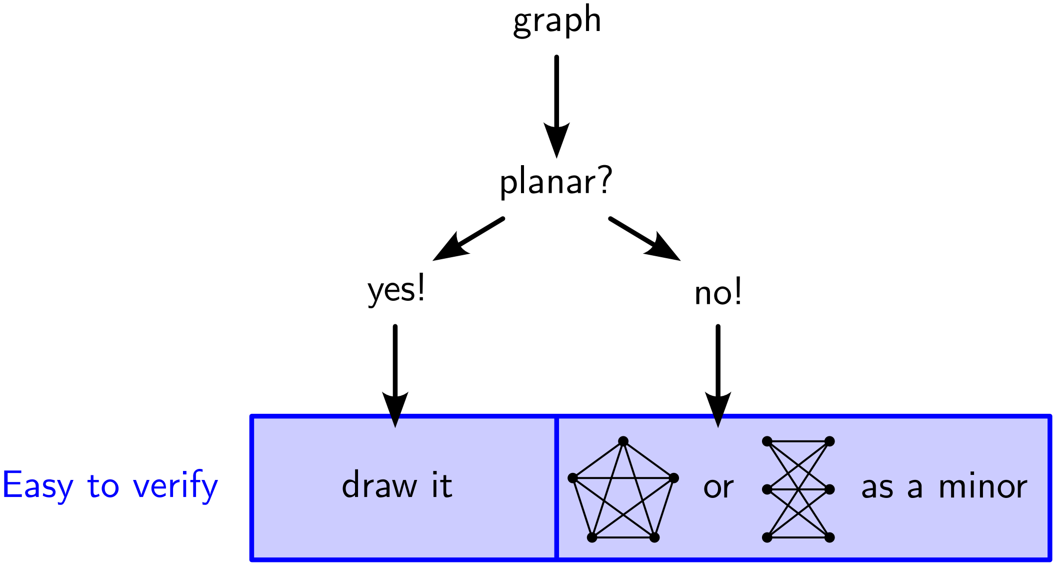
- G is planar;
- G contains neither K_5 nor $K_{3,3}$ as a minor.



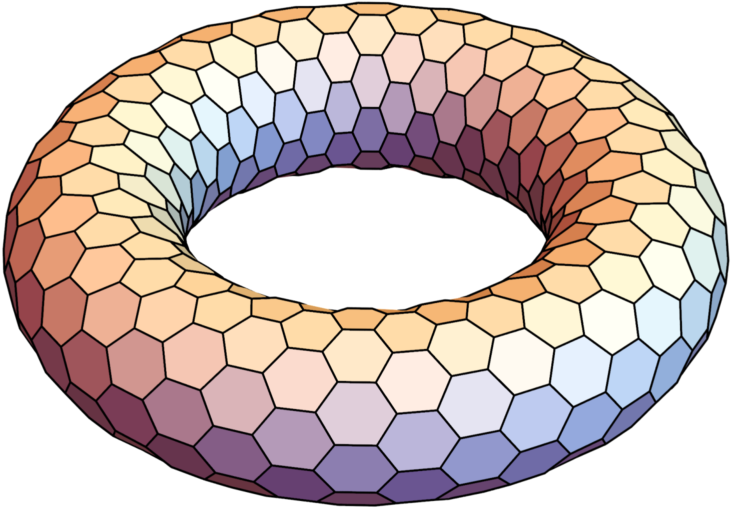
K_5



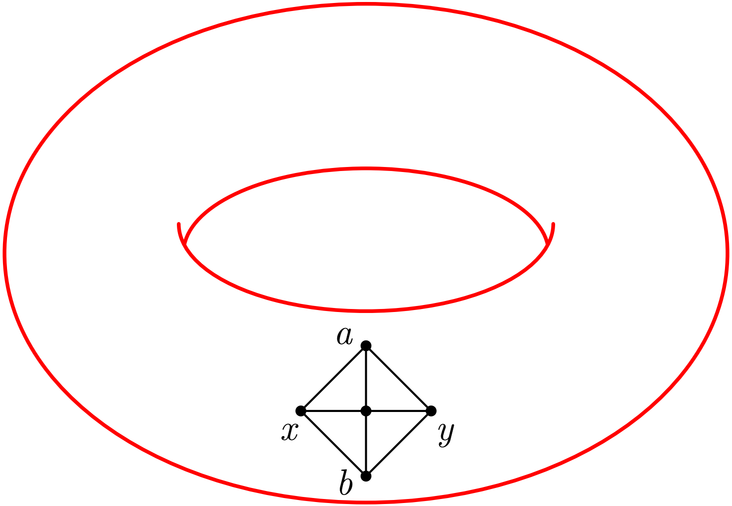
$K_{3,3}$


graph
planar?
yes!
no!
Easy to verify
draw it
or
as a minor

Is there a Kuratowski-type theorem for the torus?



Is there a planar drawing of K_5 on the torus?



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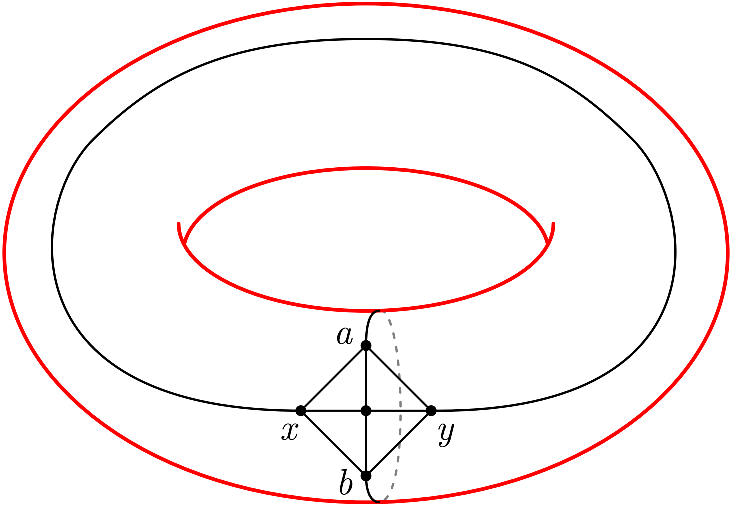
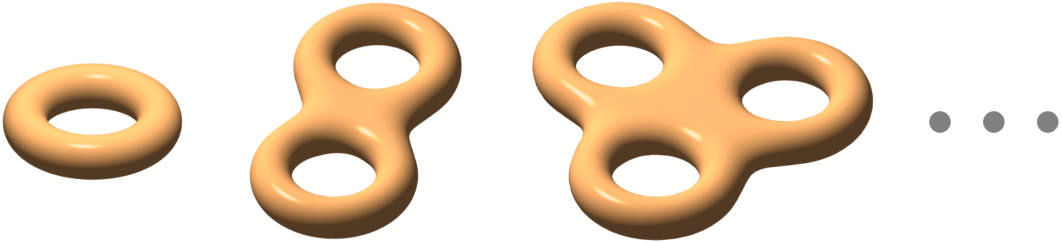
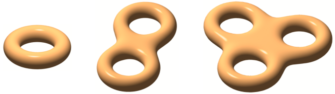


Are there Kuratowski-type theorems for other surfaces?



Conjecture (allegedly Wagner, 1960s)

For **every** graph-property $\mathcal{P}$ that is closed under taking minors

(e.g. being planar or admitting a drawing on  $\cdots$)

Conjecture (allegedly Wagner, 1960s)

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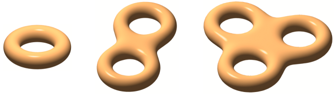
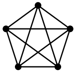
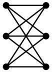
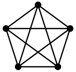
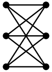
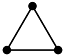
(e.g. being planar or admitting a drawing on  $\cdots$)

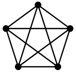
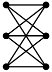
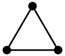
there exist **finitely many** graphs $X_1, \ldots, X_k$ such that the following assertions are equivalent:

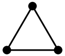
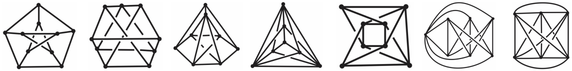
- G exhibits the property $\mathcal{P}$;
- G contains none of the graphs $X_1, \ldots, X_k$ as a minor.

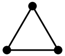
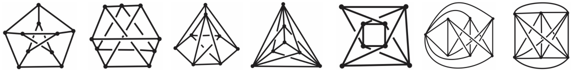
minor-closed graph-property $\mathcal{P}$	excluded minors $X_1, \ldots, X_k$
planar	 

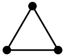
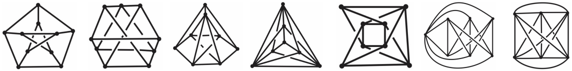
minor-closed graph-property $\mathcal{P}$	excluded minors $X_1, \ldots, X_k$
planar	
forest	

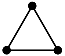
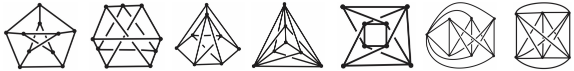
minor-closed graph-property $\mathcal{P}$	excluded minors $X_1, \ldots, X_k$
planar	 
forest	

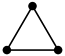
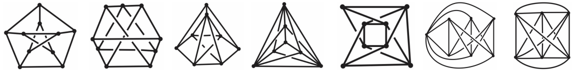
minor-closed graph-property $\mathcal{P}$	excluded minors $X_1, \ldots, X_k$
planar	 
forest	
linkless	

minor-closed graph-property $\mathcal{P}$	excluded minors $X_1, \ldots, X_k$
planar	
forest	
linkless	

minor-closed graph-property $\mathcal{P}$	excluded minors $X_1, \ldots, X_k$
planar	
forest	
linkless	
planar after deleting $\leqslant 1$ vertex	

minor-closed graph-property $\mathcal{P}$	excluded minors $X_1, \ldots, X_k$
planar	
forest	
linkless	
planar after deleting $\leqslant 1$ vertex	???

minor-closed graph-property $\mathcal{P}$	excluded minors $X_1, \ldots, X_k$
planar	
forest	
linkless	
planar after deleting $\leqslant 1$ vertex	$??? \geqslant 157$

minor-closed graph-property $\mathcal{P}$	excluded minors $X_1, \ldots, X_k$
planar	
forest	
linkless	
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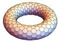
torus 	

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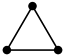
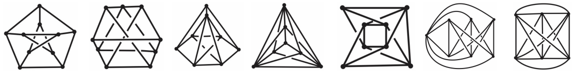
forest	
linkless	
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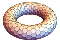
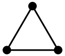
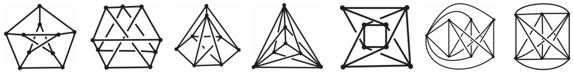
torus 	???

minor-closed graph-property $\mathcal{P}$	excluded minors $X_1, \dots, X_k$
planar	
forest	
linkless	
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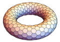
torus 	$??? \geqslant 17,523$

Conjecture (allegedly Wagner, 1960s)

For **every** minor-closed graph-property $\mathcal{P}$ there exist **finitely many** graphs $X_1, \ldots, X_k$ such that the following assertions are equivalent:

- G exhibits the property $\mathcal{P}$;
- G contains none of the graphs $X_1, \ldots, X_k$ as a minor.

Neil Robertson

Paul Seymour

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1983–2004

20 papers

$>$ **500 pages**

Paul Seymour

Corollary. For every minor-closed graph-property there exists an efficient (cubic time) algorithm for testing whether a given graph exhibits the property.

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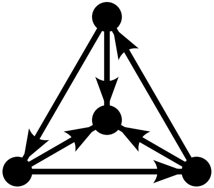
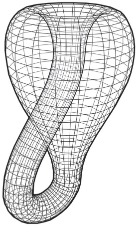


Active research

- Graph-Minor Theorem for matroids (write-up phase)
- Graph-Minor Theorem for directed graphs
- Given an explicit $\mathcal{P}$, find $X_1, \ldots, X_k$ explicitly
- Algorithms to compute $X_1, \ldots, X_k$ given $\mathcal{P}$



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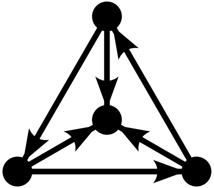

- G exhibits the property $\mathcal{P}$;
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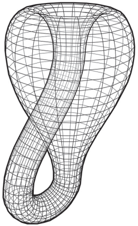

Thank you!