Canonical decompositions of 3-connected graphs

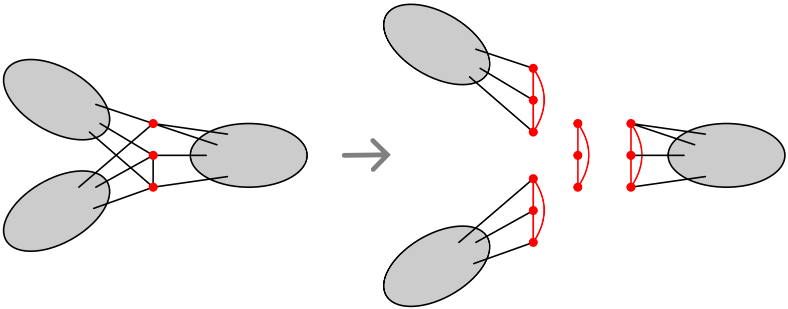
Jan Kurkofka

Joint work with Johannes Carmesin

SEG workshop on Combinatorics, Graph Theory and Algorithms

Problem: Decompose k-con'd G along k-separators into pieces that are $(k+1)$-con'd or 'basic'.

Decomposing G along a k-separator:



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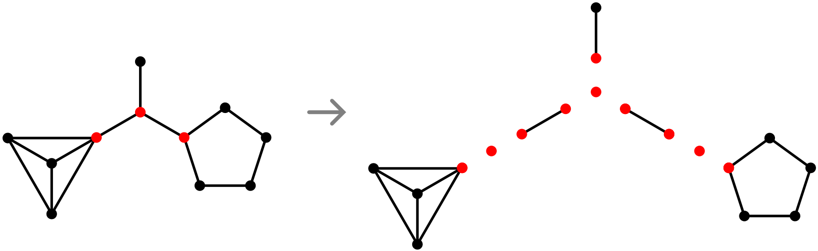

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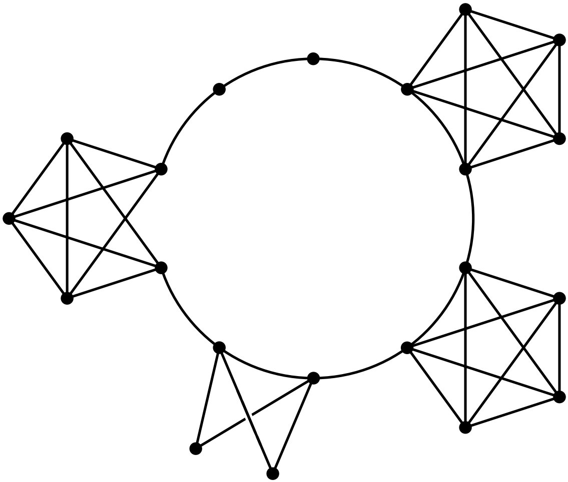

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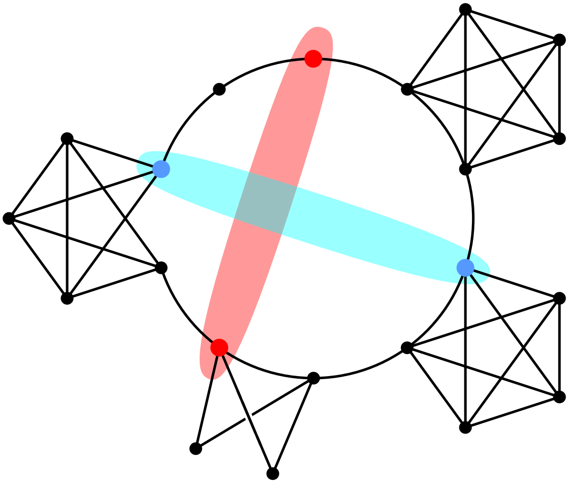

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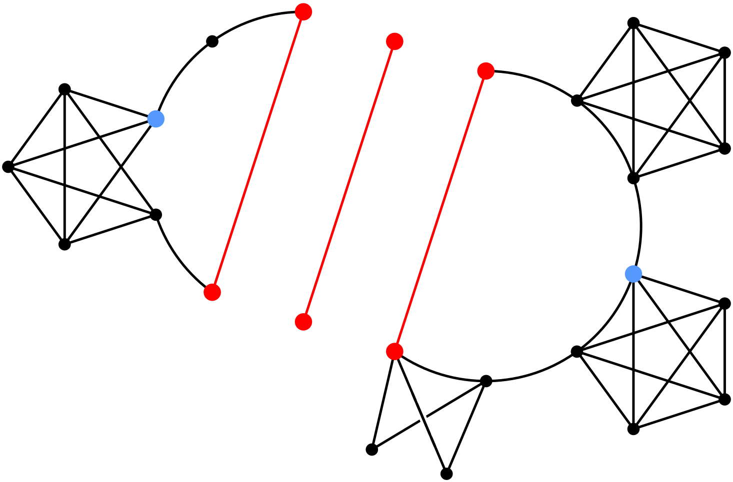
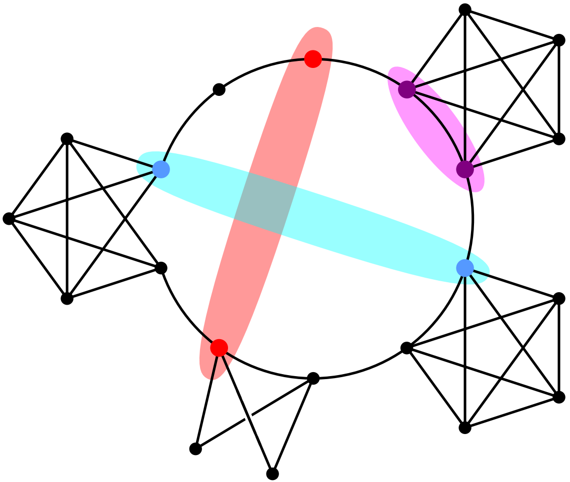


Two k-separators *cross* if they separate each other;

otherwise they are *nested*.



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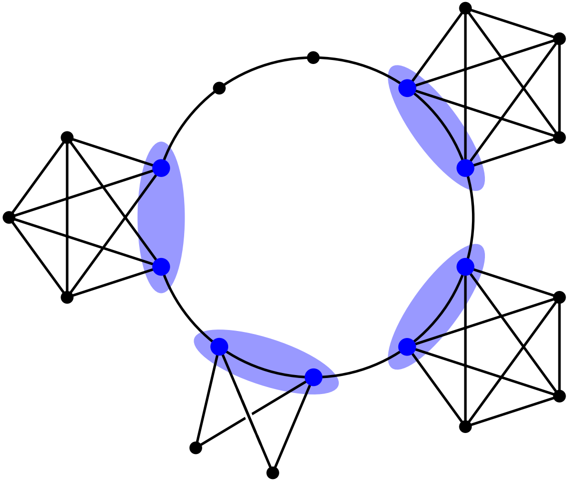

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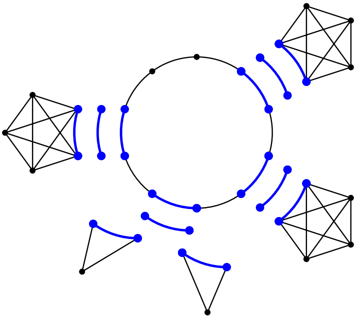
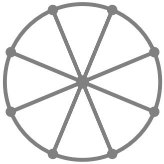


Theorem (Tutte 66), SPQR-trees

Every 2-con'd G decomposes along its totally-nested 2-separators into 3-con'd graphs, cycles and K_2's.

Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into 4-con'd graphs, wheels and K_3's.



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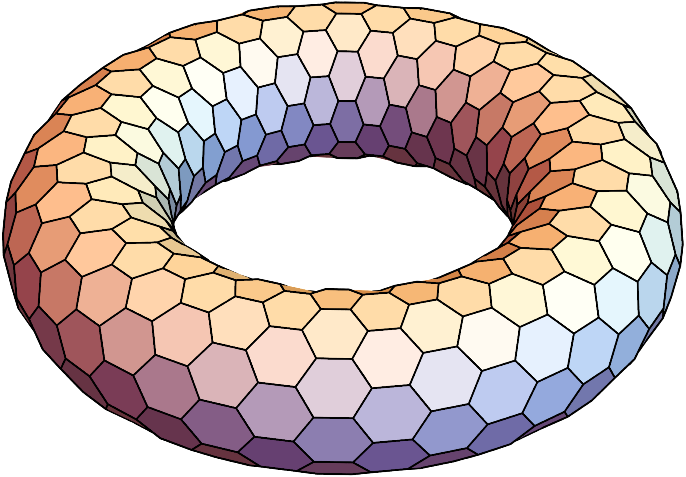

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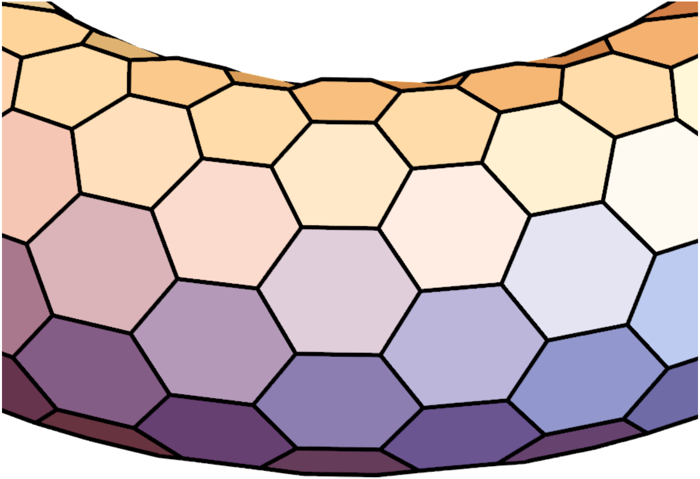

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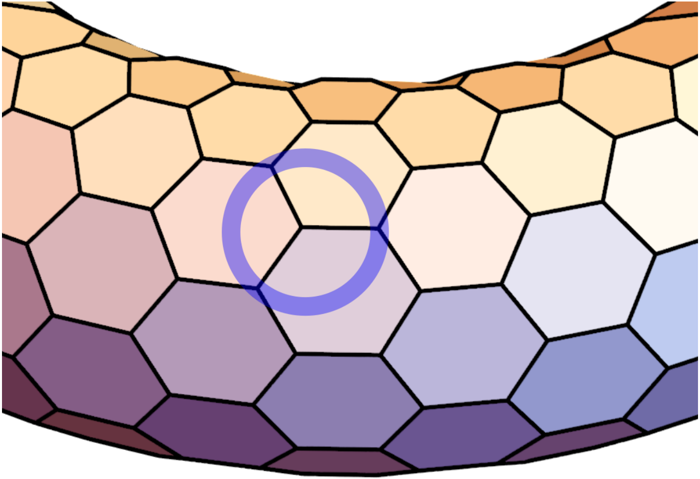

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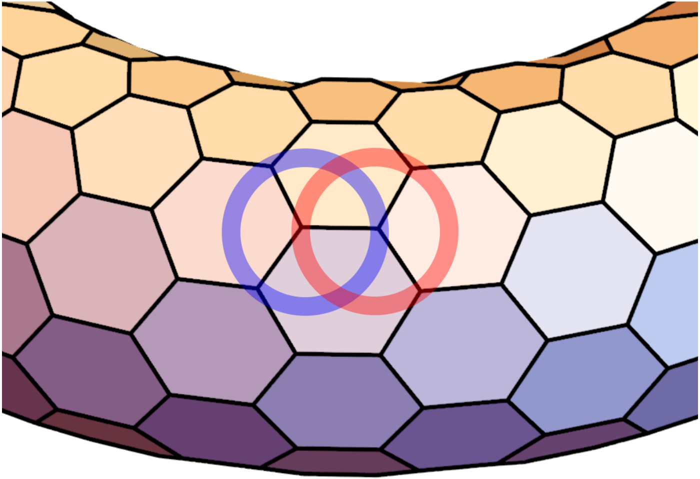

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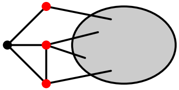
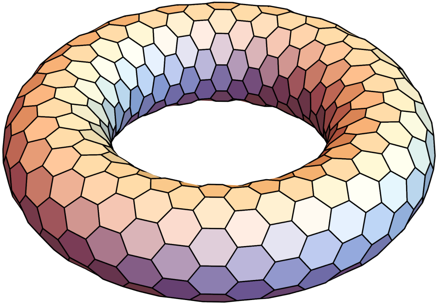
Every 3-con'd G decomposes along its totally-nested 3-separators into quasi 4-con'd graphs, wheels and K_3's.

$\downarrow$

3-con'd, > 4 vertices, every 3-separator has form 



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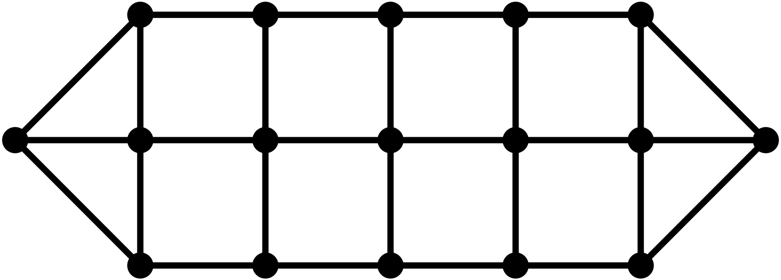

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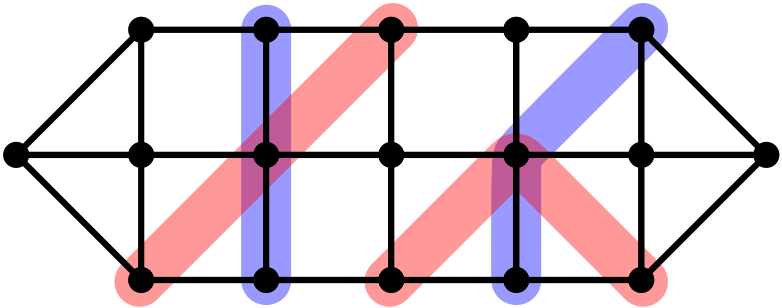

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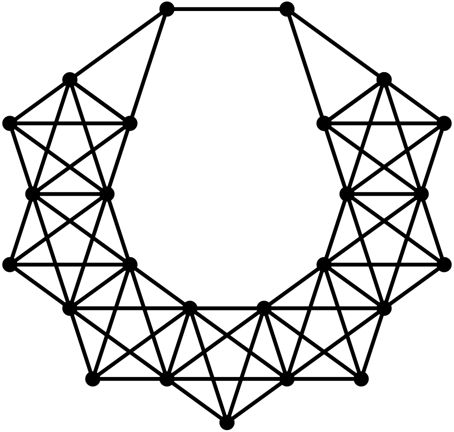

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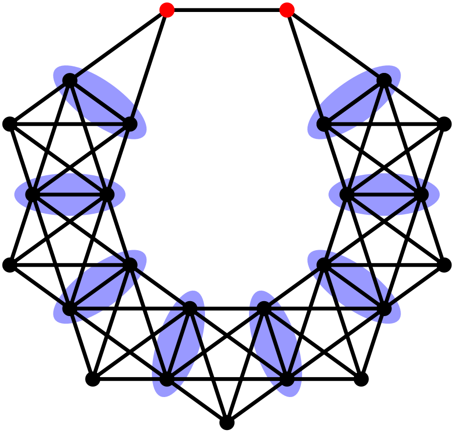

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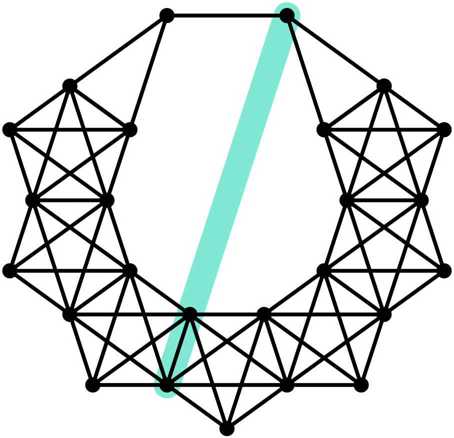

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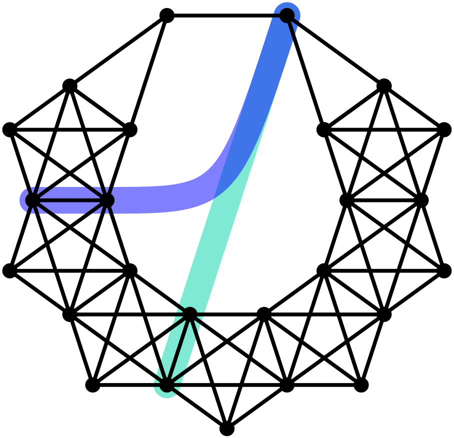

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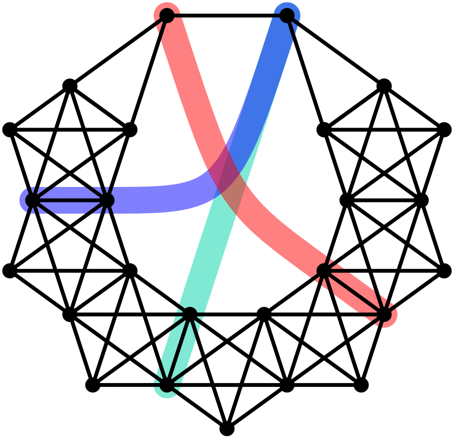

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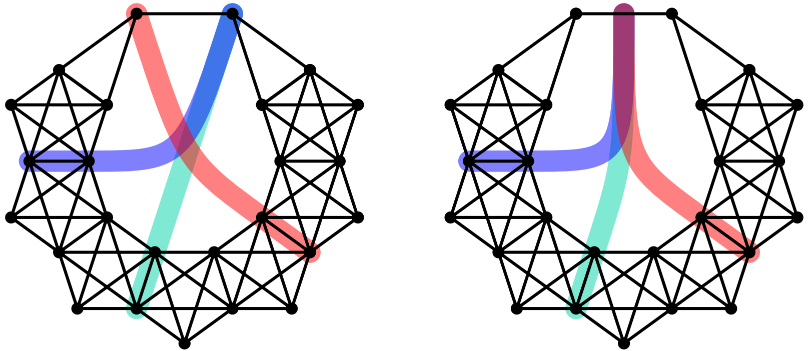

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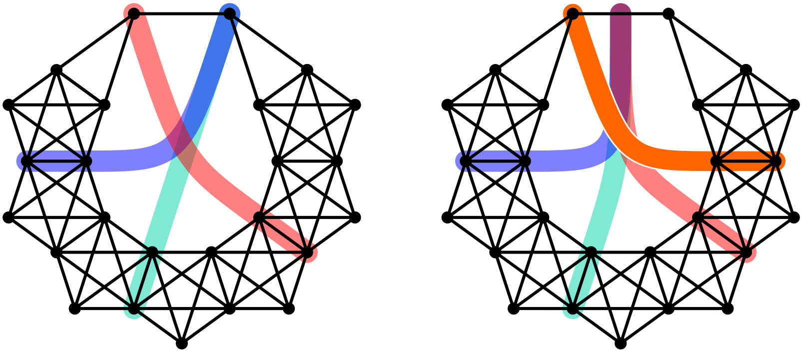

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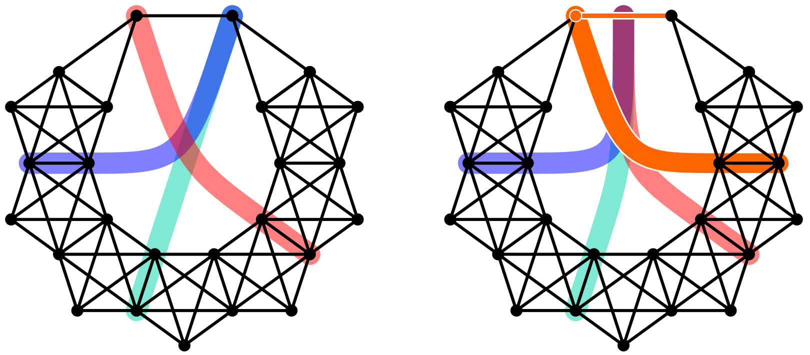

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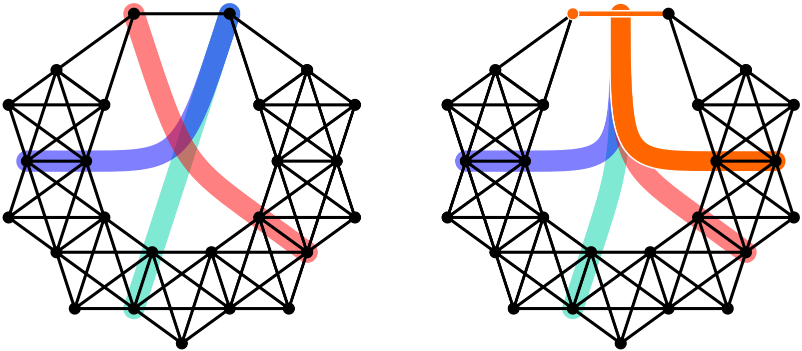
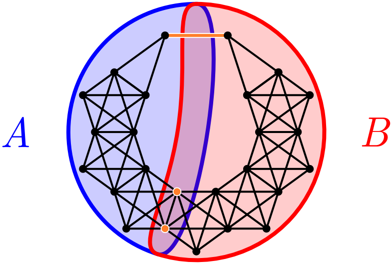
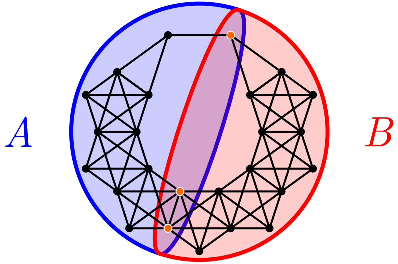




mixed-separation of G: $\{A, B\}$ with $A \cup B = V(G)$ and both $A \setminus B$ and $B \setminus A$ nonempty

separator of $\{A, B\}$: $(A \cap B) \cup E(A \setminus B, B \setminus A)$

tri-separation of G: mixed-sep'n $\{A, B\}$ with $|\text{sep'r}| = 3$ and every vx in $A \cap B$ has two neighb's in $G[A]$ and in $G[B]$



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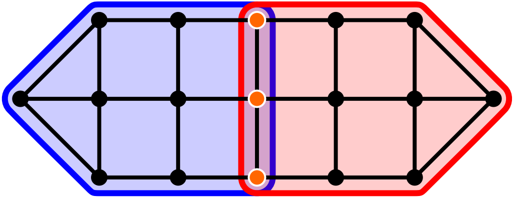

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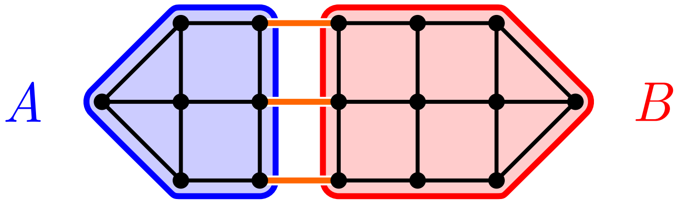

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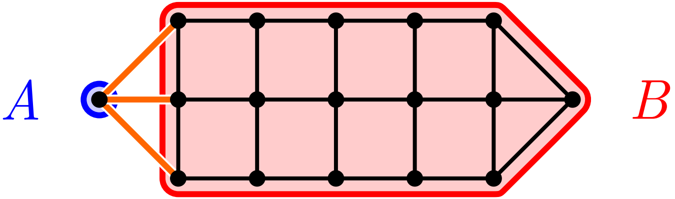
trivial



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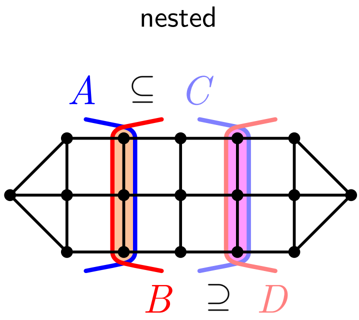
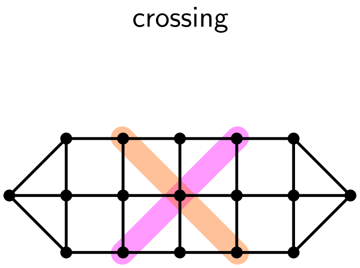
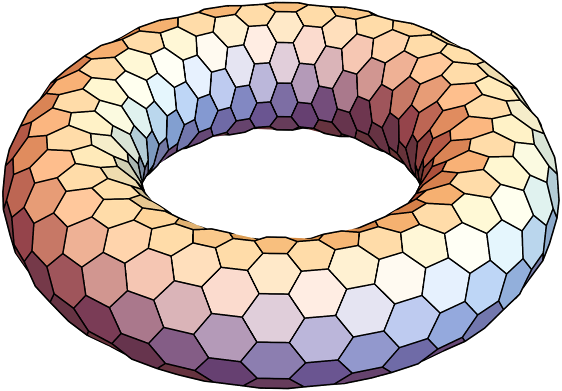
$\{A, B\}$ and $\{C, D\}$ are *nested* if $A \subseteq C$ and $B \supseteq D$ after possibly switching A with B or C with D; otherwise they *cross*.





totally-nested nontrivial tri-separations



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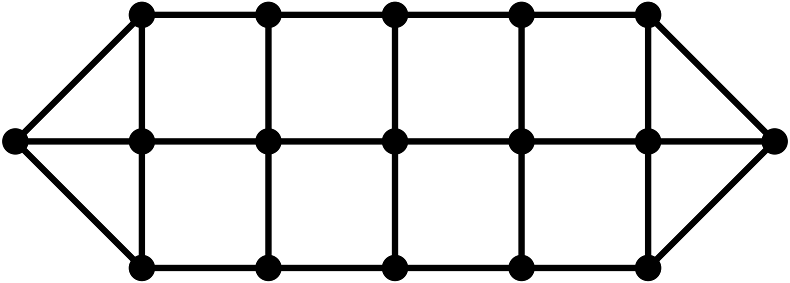

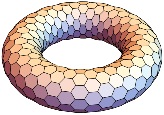

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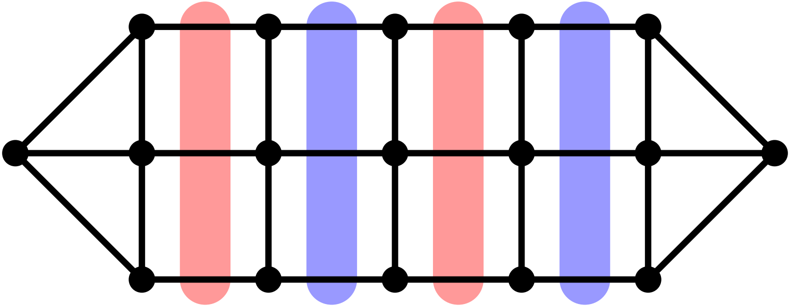

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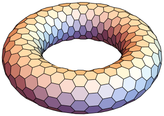

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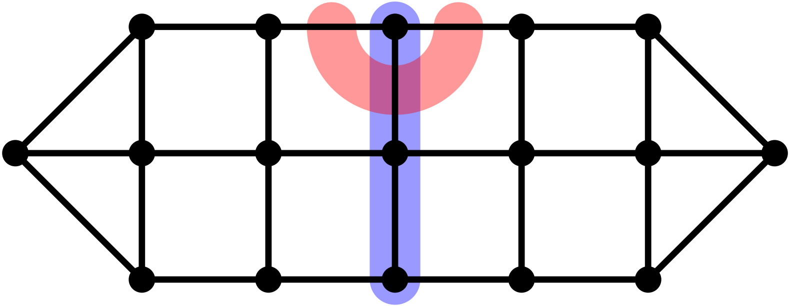

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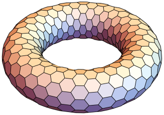

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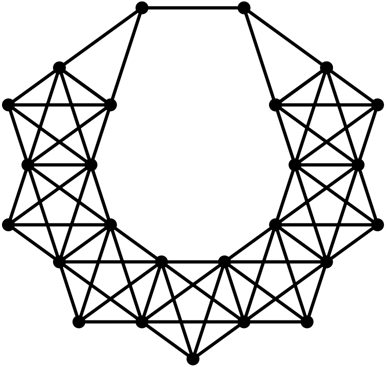

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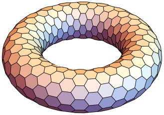

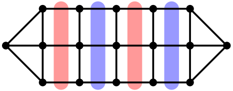

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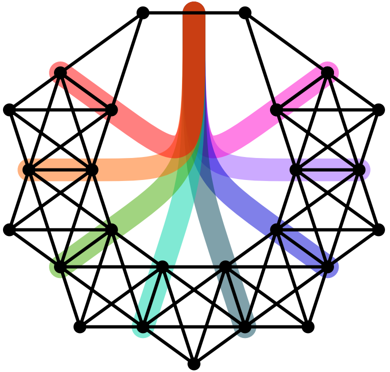

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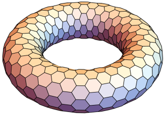
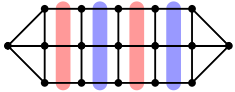
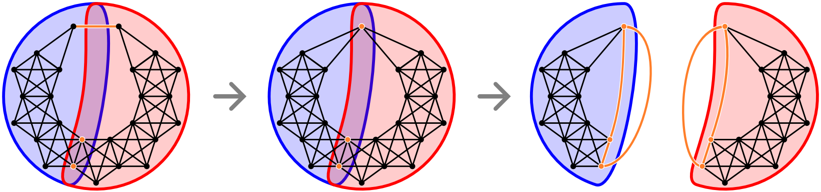
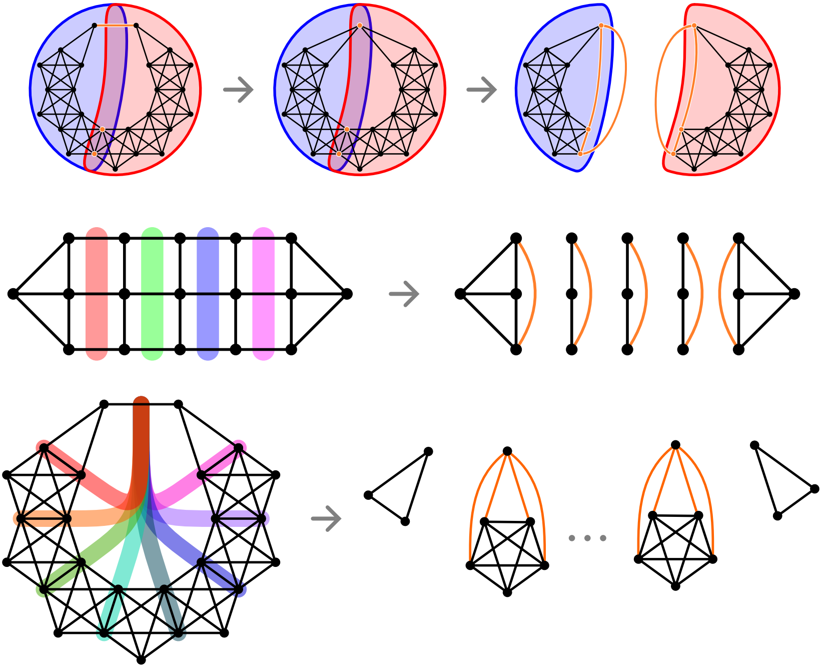




Decomposing along a tri-separation



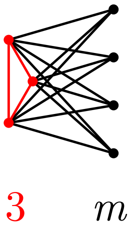


Main result (Carmesin & K. 23)

Every 3-con'd G decomposes along its totally-nested nontrivial tri-separations into minors of G that are

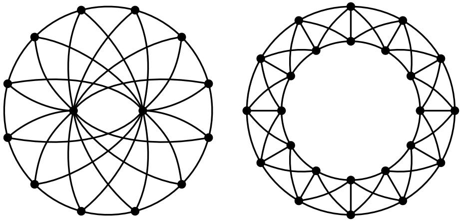
- quasi 4-con'd
- wheels
- thickened $K_{3,m}$

or $G = K_{3,m}$ ($m \geqslant 0$).

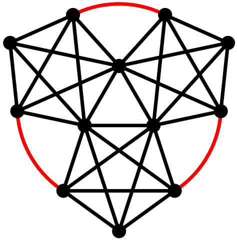


Theorem (K. & Tim Planken 24+)

4-connected graphs decompose along totally nested tetra-separations.



Application: Connectivity Augmentation from 0 to 4



Theorem (Carmesin & Sridharan 23+)

$\exists$ FPT-algorithm with runtime $C(\ell) \cdot \mathrm{Poly}(\,|V(G)|\,)$ and

Input: Graph G, $\ell \in \mathbb{N}$ and $F \subseteq E(\,\overline{G}\,)$

Output: No, or $\leqslant \ell$-sized $X \subseteq F$ such that $G + X$ is 4-con'd

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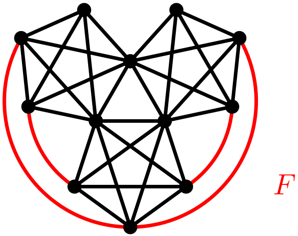

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Open: Extend the main result to *all* k.

Open: Directed graphs?

$k = 1$: Bowler, Gut, Hatzel, Kawarabayashi, Muzi, Reich 23

$k \geqslant 2$: ???

Tri-separation

Mixed-sep'n $\{A, B\}$ with $|\text{sep'r}| = 3$ such that every vx in $A \cap B$ has two neighb's in $G[A]$ and in $G[B]$.

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Open

Extend to $k \geqslant 5$. Tutte & tri-separations for digraphs.

arXiv: 2304.00945 Slides: jan-kurkofka.eu

Thank you!